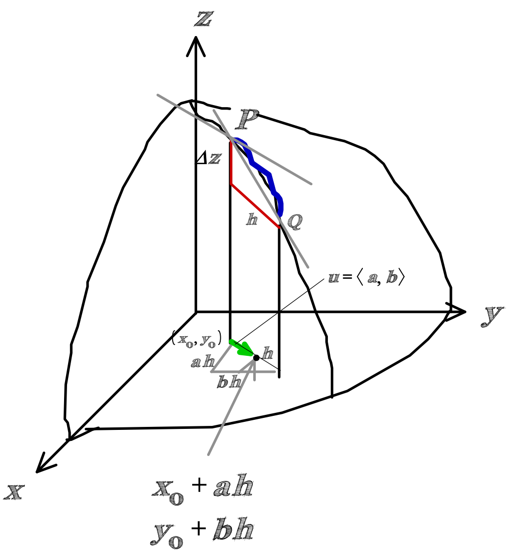


Imagine our goal is to find the slope along the blue line shown,along the surface. If we let point Q move towards P, and divide the red segment parallel to the xy plane into the red segment parallel to the z axis, we will get a slope value.

We can write it as $\frac{\Delta z}{h}$. Observe that the slice across the surface is just a curve, so we can model it with a one variable function , call it g(h). h represents how far you've gone along the green unit vector shown in the picture. Using parametric equations, we can also represent the segment in the xy plane as

$x = x_0 + a \cdot h,\ y = y_0 + b \cdot h \qquad u = \langle a, b \rangle$, unit vector

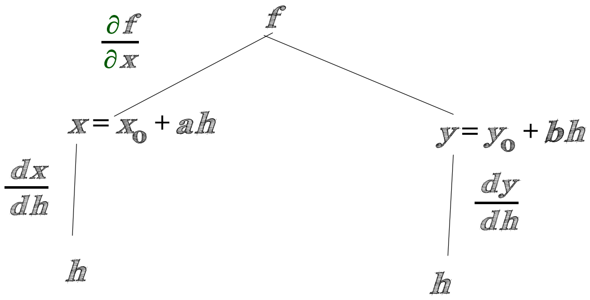
The slope at P , where h=0, can be written then as

$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$. That is, imagine the secant line PQ move so it becomes the tangent line at P.

We can write $g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} = D_u f(x, y)$ This says that the rate of change of f, in the direction of the unit vector u, is given by the expression above.

u must be a unit vector because if not, then $\sqrt{(ah)^2 + (ab)^2} = h\sqrt{a^2 + b^2} \neq h$ so $\frac{\Delta z}{h}$ would become $\frac{\Delta z}{h\sqrt{a^2 + b^2}}$, which is not the correct slope.

By the chain rule, we can also write $g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b = f_x(x, y)a + f_y(x, y)b$



From this, setting h=0, we get the point (x_0, y_0), so we have

$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$

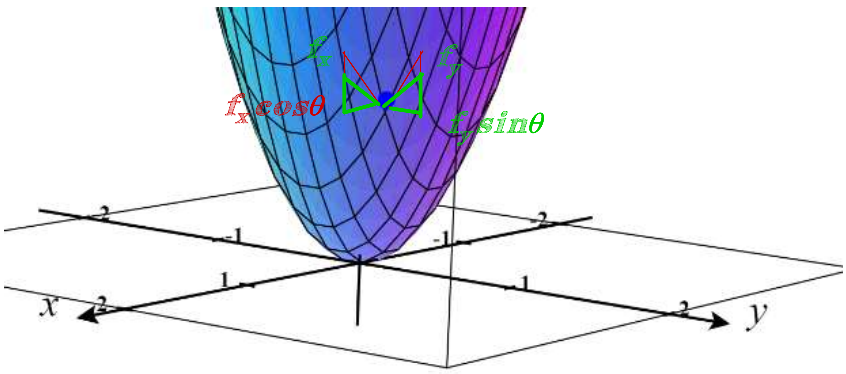
If the green vector in the picture makes an angle θ with the positive x axis, we can write it as $\langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$.

Thus, we get

$D_u f(x, y) = f_x(x, y)\cos\theta + f_y \sin(\theta)$

$= \langle f_x, f_y \rangle \bullet \langle \cos\theta, \sin\theta \rangle$

The vector $\langle f_x, f_y \rangle$ is called the gradient. It contains the partials . Since $\|\langle \cos\theta, \sin\theta \rangle\| = 1$, the direction derivative is just the scalar projection of the gradient onto the unit vector $\langle \cos\theta, \sin\theta \rangle$. In other word cos θ takes a fraction of f_x and adds that fraction of f_x to the fraction that $\sin\theta$ takes of f_y.



Example: Find the slope on $x^2 + y^2$ in the direction of $\langle 1, 1 \rangle$ at the point (1,1).

$D_u f(x, y) = \langle 2x, 2y \rangle \bullet \frac{\langle 1, 1 \rangle}{\sqrt{1^2 + 1^2}}$

$= \langle 2x, 2y \rangle \bullet \frac{1}{\sqrt{2}} \langle 1, 1, \rangle$

$= \frac{2x}{\sqrt{2}} + \frac{2y}{\sqrt{2}}$

$= \frac{2}{\sqrt{2}}(x + y)$

$D_u f(1, 1) = \frac{2}{\sqrt{2}}(1 + 1) = \frac{4}{\sqrt{2}} \approx 2.8$

Notice that $2(1) \cdot \cos(45) = 2 \cdot \frac{1}{\sqrt{2}} = 2 \cdot 0.7$ =70% of f_x

$2(1)\ \sin(45) = 2 \cdot \frac{1}{\sqrt{2}} = 2 \cdot 0.7 =$ 70 % of f_y

Find the directional derivative of $f(x,y)=x^3+3xy+y^2$ at the point(1,2), towards (3,4).

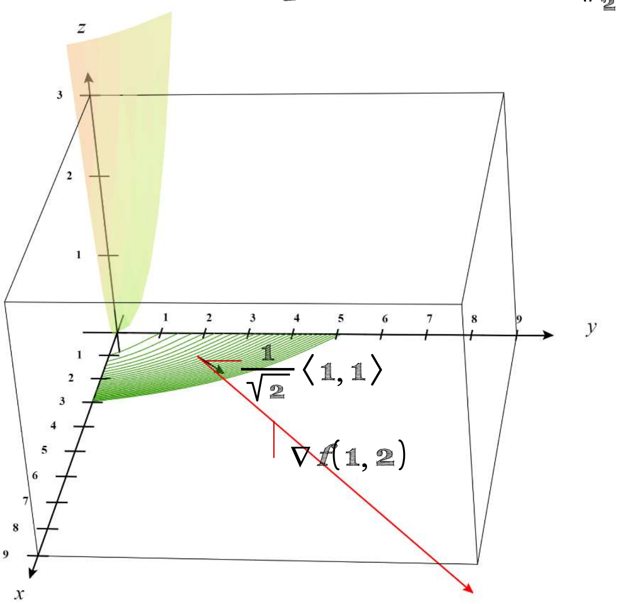
$$\nabla f(x,y)=\left\langle \frac{\partial}{\partial x}(x^3+3xy+y^2),\ \frac{\partial}{\partial y}(x^3+3xy+y^2)\right\rangle=\langle 3x^2+3y, 3x+2y\rangle$$

$\nabla f(1,2)=\langle 3\cdot 1^2+3\cdot 2, 3\cdot 1+2\cdot 2\rangle=\langle 9,7\rangle$ [This is the gradient vector at the point (1,2)]

unit vector from (1,2) to (3,4): $\dfrac{\langle 3-1, 4-2\rangle}{\sqrt{(3-1)^2+(4-2)^2}}=\dfrac{\langle 2,2\rangle}{\sqrt{2^2+2^2}}=\dfrac{1}{\sqrt{4+4}}\langle 2,2\rangle=\dfrac{1}{2\sqrt{2}}\langle 2,2\rangle=\dfrac{1}{\sqrt{2}}\left\langle \dfrac{2}{2},\dfrac{2}{2}\right\rangle$

$$=\frac{1}{\sqrt{2}}\langle 1,1\rangle$$

Directional Derivative: $D_u f(1,2)=\nabla f(1,2)\bullet\frac{1}{\sqrt{2}}\langle 1,1\rangle=\frac{1}{\sqrt{2}}\langle 9,7\rangle\bullet\langle 1,1\rangle$

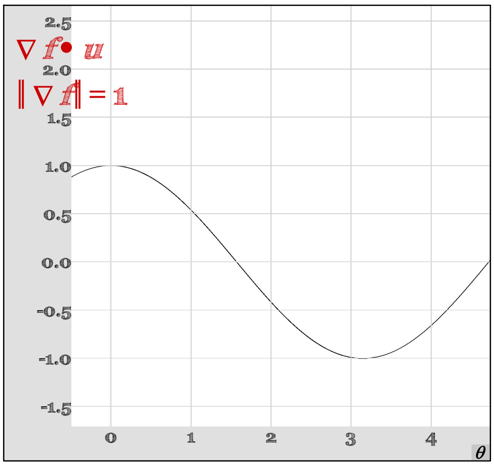


$$=\frac{1}{\sqrt{2}}[9\cdot 1+7\cdot 1]$$
$$=\frac{1}{\sqrt{2}}(9+7)$$
$$=\frac{1}{\sqrt{2}}(16)$$
$$\approx 11.3$$

To maximize the value of the directional derivative, we can write

$D_u f(x,y)=\nabla f\bullet u=\|\nabla f\|\|u\|\cos(\theta)=\|\nabla f\|\cos(\theta)$ since $\|u\|=1$ (u is a unit vector)

For the sake of illustration, set $\|\nabla f\|=1$, so we get $\nabla f\bullet u=\cos(\theta)$. Thus, we get

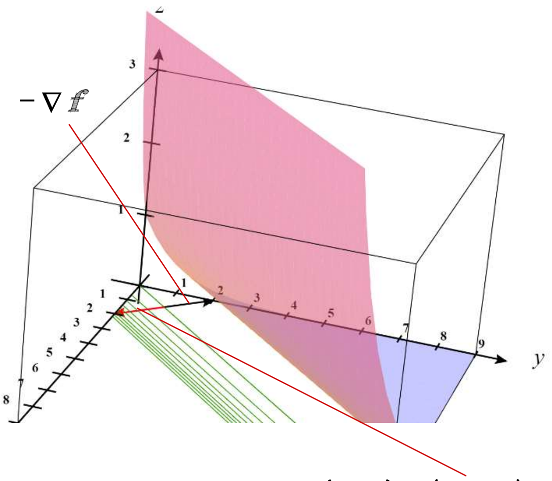


the maximum value of D_u f(x,y) occurs when the angle between the unit vector and ∇f is $\theta=0$.

We can also tell that the minimum value of $D_u f(x,y)$ occurs when $\theta=\pi$.

For $f(x,y)=e^{x-y}$, find the direction of maximum increase at the point (1,1) and the direction of maximum decrease.

$$\nabla f(x,y)=\left\langle \frac{\partial}{\partial x}e^{x-y},\ \frac{\partial}{\partial y}e^{x-y}\right\rangle=\left\langle e^{x-y}\frac{\partial}{\partial x}(x-y),\ e^{x-y}\frac{\partial}{\partial y}(x-y)\right\rangle$$
$$=\langle e^{x-y}(1), e^{x-y}(-1)\rangle$$
$$=\langle e^{x-y}, -e^{x-y}\rangle$$



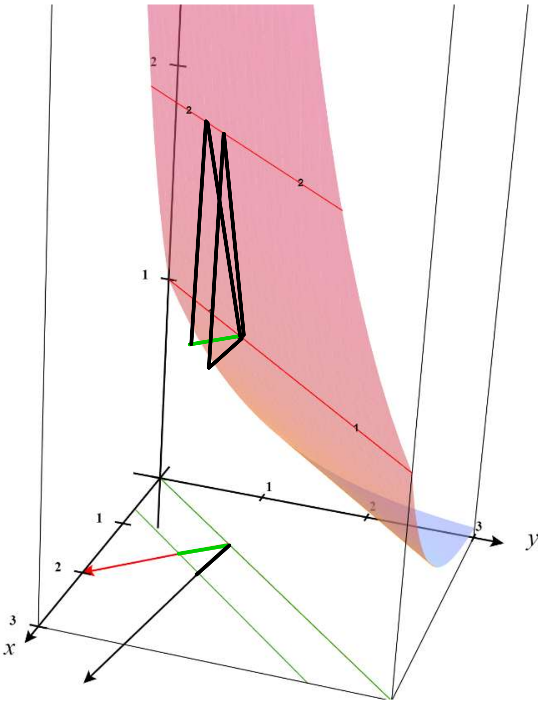
$\nabla f(1,1)=\langle e^{1-1}, -e^{1-1}\rangle=\langle e^0, -e^0\rangle=\langle 1,-1\rangle$

$\langle 1,-1\rangle$ is the direction of maximum increase.

$-\langle 1,-1\rangle=\langle -1,1\rangle$ is the direction of maximum decrease.

The rate of maximum increases is the magnitude of the gradient vector: $\|\nabla f(1,1)\|=\sqrt{1^2+(-1)^2}=\sqrt{1+1}=\sqrt{2}$

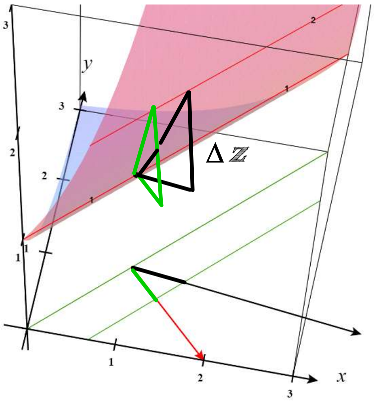
$\nabla f(1,1)=\langle 1,-1\rangle$ Notice it shows the direction of maximum increase and is perpendicular to the level curve.



Since the green segment is bigger in lenth then the black segment in the plane, and Δz is the same for both triangles shown,

we get that $\dfrac{\Delta z}{\text{length of green segment}} > \dfrac{\Delta z}{\text{length of black segment}}$

The black segment is longer because it's slanted when it goes from level curve to level curve.



Notice that Δz is the same for both, since the level curves go from 1 to 2 for both.

You see the green segment is shorter, so

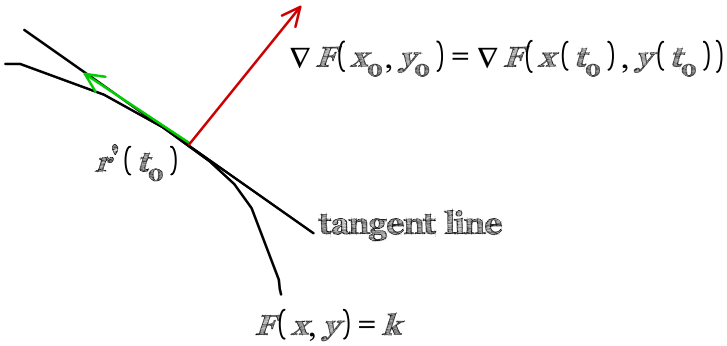
$\dfrac{\Delta z}{\text{green length}} > \dfrac{\Delta z}{\text{black length}}$

Notice that if we have a level curve from $F(x, y) = k$, the equation of the tangent line can be written as $r'(t) = \langle x'(t), y'(t)\rangle$. At the point of tangency, we must have $F(x(t), y(t)) = k$ true, so by the chain rule, we get $\dfrac{\partial F}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial F}{\partial y}\dfrac{dy}{dt} = 0$, which in vector form can be written as

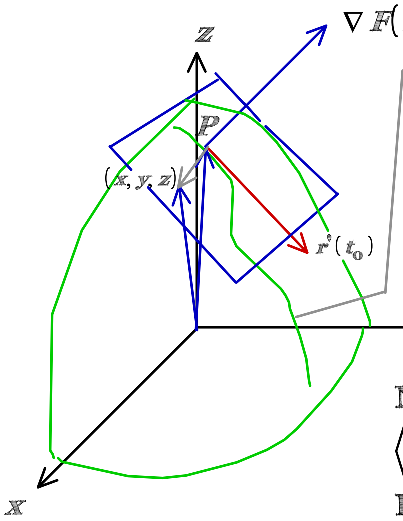
$$\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right\rangle \bullet \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle = 0$$

$$\nabla F \bullet r'(t) = 0$$

But this says that the gradient vector and the r'(t)/ level curves are perpendicular.



Tangent Planes to Level Surfaces:



This path along the surface can be represented by
$r(t) = \langle x(t), y(t), z(t) \rangle$
and a tangent vector can be represented by $r'(t) = \langle x'(t), y'(t), z'(t) \rangle$

The surface itself can also be imagined as a level surface of a function of three variables. $F(x, y, z) = k$. Along the path, x=x(t),y=y(t),z=z(t), so we get $F(x(t), y(t), z(t)) = k$. Notice that we can differentiate this using the chain rule to get $\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$

Notice that this can be written as a dot product:
$\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \bullet \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = 0$
But this says the gradient vector is perpendicular to r'.

At the specific value of t=t$_0$, we then get $\nabla F(x(t_0), y(t_0), z(t_0)) \bullet r'(t_0) = 0$
Since ∇F is normal to r'(t), we can create the equation of a tangent plane through the point $P(x_0, y_0, z_0)$ and some other point (x,y,z). Thus we get, using $\nabla F(x_0, y_0, z_0)$ as the direction vector,
$\nabla F(x_0, y_0, z_0) \bullet \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$\langle F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0) \rangle \bullet \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$

Example: Find the tangent plane to $x^2 + y^2 + z^2 = 3$ at the point (1,1,1). First note that $1^2 + 1^2 + 1^2 = 1 + 1 + 1 = 3$ checks with the right side.
Form F(x,y,z)= $x^2 + y^2 + z^2 - 3$
Form the derivatives:

$F_x(x, y, z) = 2x$
$F_y(x, y, z) = 2y$
$F_z(x, y, z) = 2z$

Evaluate these derivatives at the point (1,1,1)
$F_x(1, 1, 1) = 2$ Now the equation of the tangent plane is
$F_y(1, 1, 1) = 2$
$F_z(1, 1, 1) = 2$

$\langle 2, 2, 2 \rangle \bullet \langle x - 1, y - 1, z - 1 \rangle = 0$

$2(x-1) + 2(y-1) + 2(z-1) = 0$

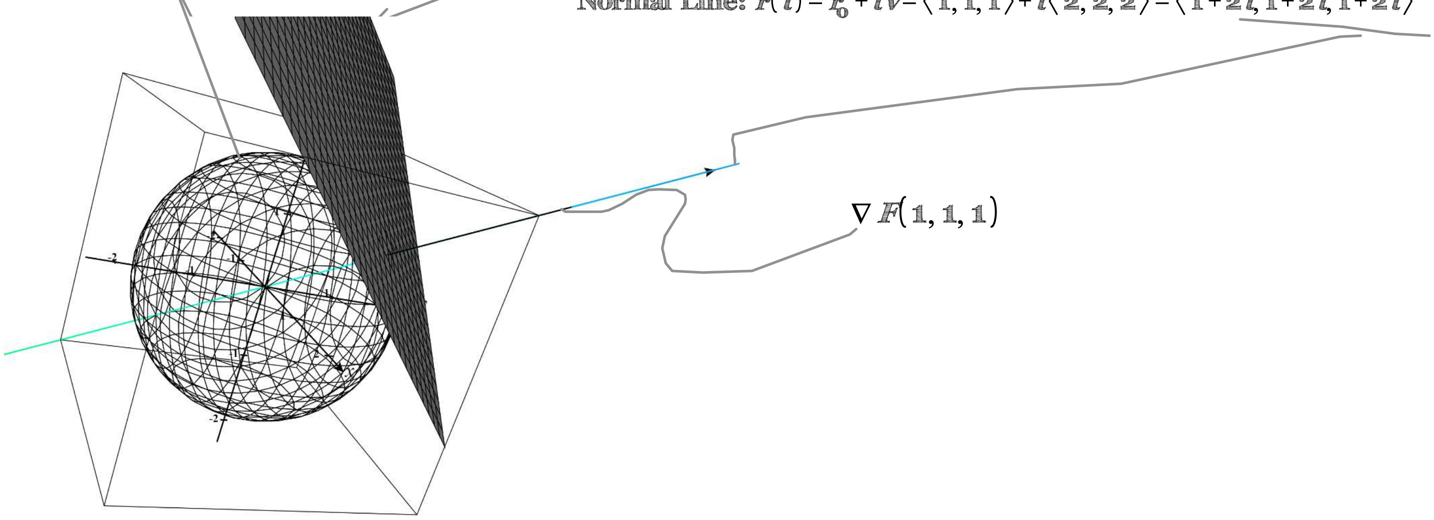
$2x - 2 + 2y - 2 + 2z - 2 = 0$

$2x + 2y + 2z - 6 = 0$

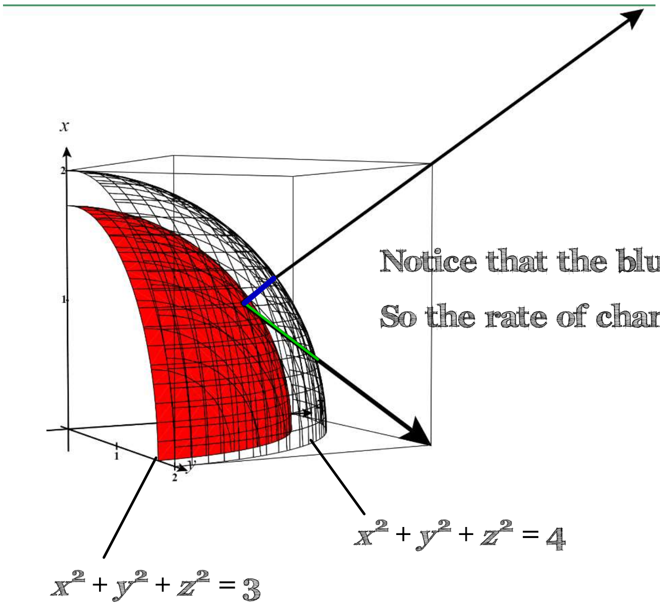
$2x + 2y + 2z = 6$

$x + y + z = 3$

Normal Line: $r(t) = r_0 + tv = \langle 1, 1, 1 \rangle + t\langle 2, 2, 2 \rangle = \langle 1 + 2t, 1 + 2t, 1 + 2t \rangle$



Note that as in the case of level curves, it's true also that the gradient vector gives the direction of fastest increase for a function F(x,y,z).



Notice that the blue segment is shorter than the green segment.

So the rate of change formed by $\dfrac{\Delta F}{\text{green segment length}} < \dfrac{\Delta F}{\text{length of blue segment}}$

Suppose the temperature in space is given by $T(x,y,z)=\dfrac{1}{x^2+y^2+z^2}$. T is in degrees Celcius and position is in meters. At the point $(1,1,1)$, in which direction does the temperature increase the fastest and what is the rate of increase?

$$\nabla F(x,y,z)=\left\langle \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1}, \frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1}, \frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1}\right\rangle$$

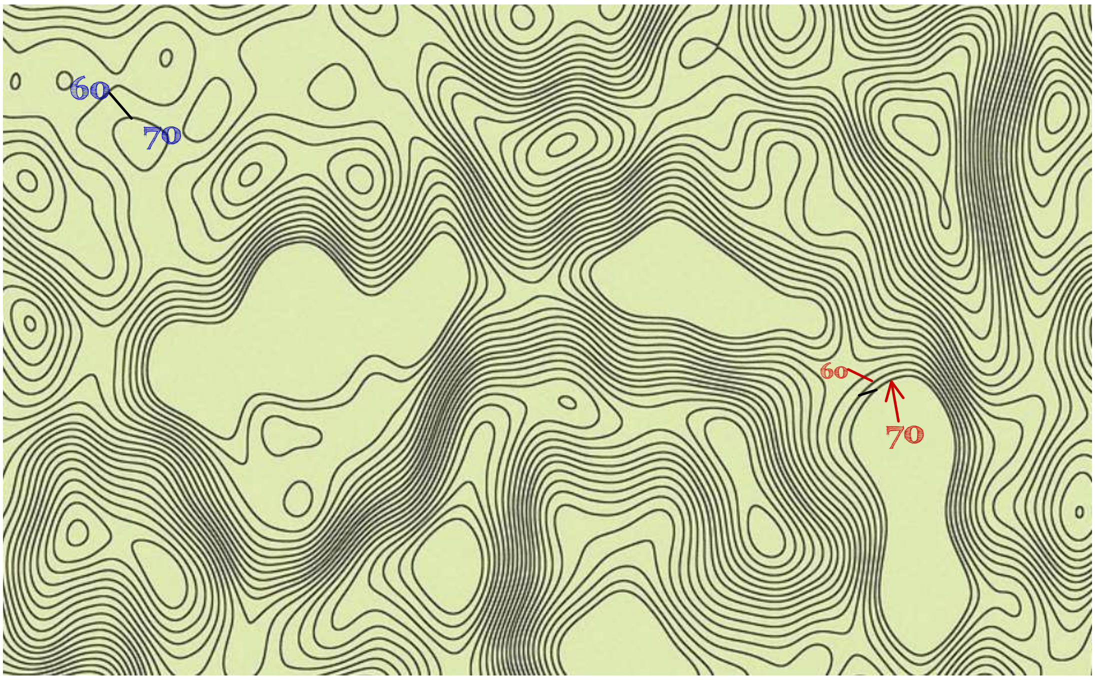
$$=\left\langle \frac{-1}{(x^2+y^2+z^2)^2}\frac{\partial}{\partial x}(x^2+y^2+z^2), \frac{-1}{(x^2+y^2+z^2)^2}\frac{\partial}{\partial y}(x^2+y^2+z^2), \frac{-1}{(x^2+y^2+z^2)^2}\frac{\partial}{\partial z}(x^2+y^2+z^2)\right\rangle$$

$$=\frac{-1}{(x^2+y^2+z^2)}\langle 2x, 2y, 2z\rangle$$

$$\nabla F(1,1,1)=\frac{-1}{1^2+1^2+1^2}\langle 2\cdot 1, 2\cdot 1, 2\cdot 1\rangle=\frac{-1}{3}\langle 2,2,2\rangle=\left\langle \frac{-2}{3}, \frac{-2}{3}, \frac{-2}{3}\right\rangle$$ direction of max. increase

Rate of Max. Increase:

$$\|\nabla F(1,1,1)\|=\sqrt{\left(\frac{-2}{3}\right)^2+\left(\frac{-2}{3}\right)^2+\left(\frac{-2}{3}\right)^2}=\sqrt{3\cdot\frac{4}{9}}=\sqrt{\frac{4}{3}}=\frac{2}{\sqrt{3}}\approx 1.15\,°\text{C/m}$$ (1.15 degrees Celcius per meter)



Imagine on the contour map the level curves represent temperature and the distance scale is that a — represents about 10 miles. 1.

Estimate D_u T according to the picture.

$$D_u\ T \approx \frac{70-60}{\text{distance}} = \frac{10}{10/2.5} = \frac{10}{4} = 2.5°/\text{mi}$$

$=$ 2.5 segments from the space between the level curves.

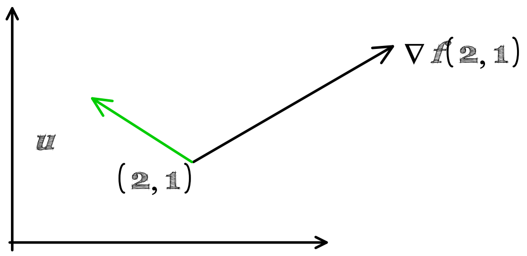
so $10/2.5 = 4$ miles

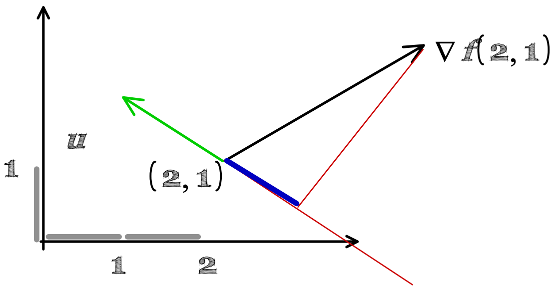
For the blue numbers above, we get $D_u T \approx \frac{70-60}{8} = \frac{10}{8} = 1.25$°C/mi

$=$ 10 miles

Shorter segment seems to be about 8 miles.

Estimate the value of $D_u f(2,1)$ using the picture below. 2.





To answer, project $\nabla f(2,1)$ onto the unit vector.

Add numbers along the axes to get a better sense of scale. It seems the length of the blue segment is about 1, but the angle between u and $\nabla f(2,1)$ is more than 90, so the scalar projection is about $-1 \approx D_u f(2,1)$

3 Where does the minimum value of $D_u f$ occur ? This is so because $\nabla f \bullet u = \|\nabla f\| \cos(\theta)$, $\|u\| = 1$

With $\|\nabla f\| = 1$, we get $\cos(\theta)$, which is minimized at $\theta = \pi$.

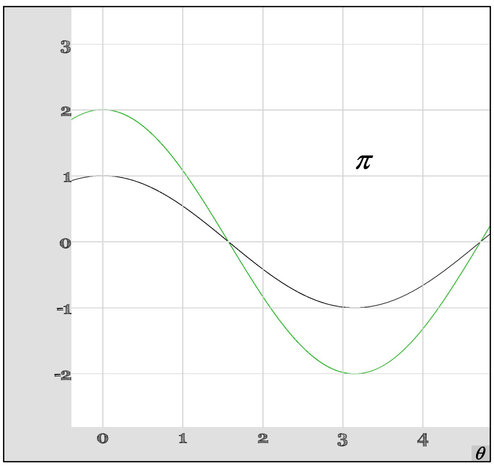
If you set $\|\nabla f\| = 2$, the location of the minimum doesn't change.

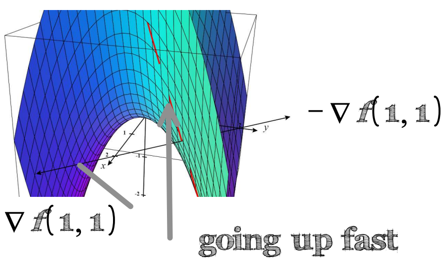
Thus we get $\nabla f \bullet u = \|\nabla f\| \cos(\pi) = -\|\nabla f\|$ This says the direction of fastest decrease is $-\nabla f$.

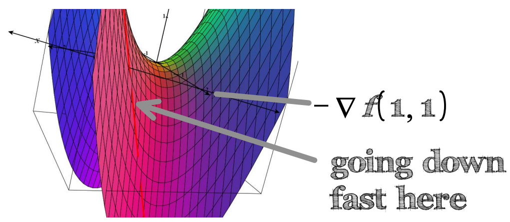
Example: $f(x,y) = x^2 - y^2$ gives us

$\nabla f(x,y) = \langle 2x, -2y \rangle$ At (1,1), we get $\nabla f(1,1) = \langle 2\cdot 1, -2\cdot 1 \rangle = \langle 2, -2 \rangle$

Direction of fastest decrease $= -\nabla f(1,1) = -\langle 2, -2 \rangle = \langle -2, 2 \rangle$







Suspect you're climbing a hill whose shape is gien by $z = 1000 - 0.004x^2 - 0.05y^2$, where x and y and are in meters and you're standing at the opint (60,40, 906) . If you walk south, will you ascend or descend? Find the rate.

$1000 - 0.004(60)^2 - 0.05(40)^2 \approx 906$ meters

Walking south tells us the direction is $-\hat{j}$ so we have to find $D_{-j}z(60, 40) = \nabla z(60, 40) \bullet \langle 0, -1 \rangle$ 4.

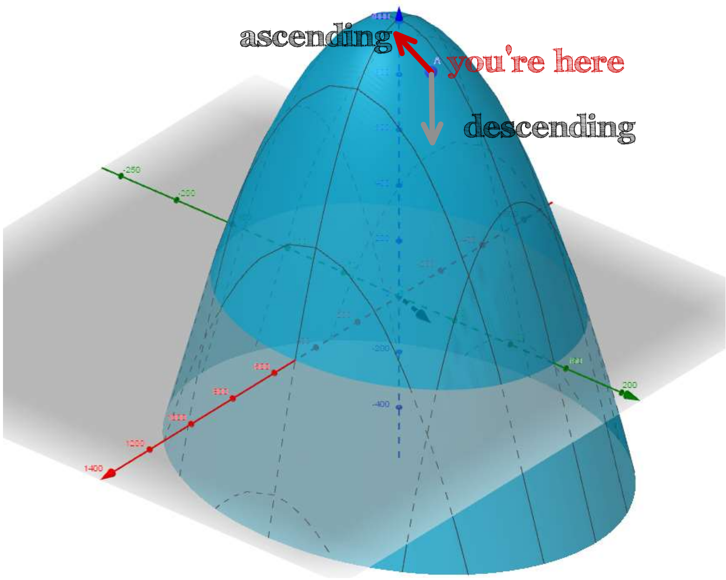
$\nabla z(x, y) = \langle -0.008x, -0.10y \rangle \Rightarrow \nabla z(60, 40) = \langle -0.008(60), -0.10(40) \rangle = \langle -0.480, -4 \rangle$

So we get $D_{south} z(60, 40) = \langle -0.480, -4 \rangle \bullet \langle 0, -1 \rangle = -0.480(0) - 4(-1) = 4$ This says you will ascend at a rate of 4 vertical meters for every horizontal meter.

If you walk northwest, that's along the vector $\langle 1, 1 \rangle$,which in unit form is $\frac{1}{\sqrt{2}}\langle 1, 1, \rangle$. So we get

$D_{northwest} z(60, 40) = \langle -0.480, -4 \rangle \bullet \frac{1}{\sqrt{2}}\langle 1, 1 \rangle = \frac{1}{\sqrt{2}}[-0.480 - 4] = \frac{1}{\sqrt{2}}(-4.480) \approx -3.168$

This says when you walk northwest, you descend by 3.168 meters per horizontal meter.



Proofs of a couple rules concering gradients: u,v functions and a,b constants 5.

$\nabla(au + bv) = a\nabla u + b\nabla v$

$$\nabla(au + bv) = \left\langle \frac{\partial}{\partial x}(au + bv), \frac{\partial}{\partial y}(au + bv) \right\rangle$$

$$= \left\langle a\frac{\partial}{\partial x}u + b\frac{\partial}{\partial x}v, a\frac{\partial}{\partial y}u + b\frac{\partial}{\partial y}v \right\rangle$$

$$= \left\langle a\frac{\partial}{\partial x}u, a\frac{\partial}{\partial y}u \right\rangle + \left\langle b\frac{\partial}{\partial x}v, b\frac{\partial}{\partial y}v \right\rangle$$

$$= a\langle u_x, u_y \rangle + b\langle v_x, v_y \rangle$$

$$= a\nabla u + b\nabla v$$

You should try

$\nabla(uv)$

$\nabla\left(\frac{u}{v}\right)$

$$\nabla u^n = \left\langle \frac{\partial}{\partial x}u^n, \frac{\partial}{\partial y}u^n \right\rangle = \left\langle nu^{n-1}\frac{\partial}{\partial x}u, nu^{n-1}\frac{\partial}{\partial y}u \right\rangle = nu^{n-1}\langle u_x, u_y \rangle = nu^{n-1}\nabla u$$

6. Find the equations of the normal line and tangent plane to $xy^2z^3 = 8$ at the point (2,2,1).

$F(x,y,z) = xy^2z^3$. Then $xy^2z^3 = 8$ is a level surface of F and $\nabla F(x,y,z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$

So $\nabla F(2,2,1)$ is a normal vector for the plane tangent to the graph at (2,2,1).Thus, an equation of the plane is $\langle 2^2 \cdot 1^3, 2\cdot 2\cdot 2\cdot 1^3, 3\cdot 2\cdot 2^2\cdot 1^2 \rangle \bullet \langle x-2, y-2, z-1 \rangle = 0$

$\langle 4, 8, 24 \rangle \bullet \langle x-2, y-2, z-1 \rangle = 0$

$4(x-2)+8(y-2)+24(z-1)=0$

divide 4 away: $(x-2)+2(y-2)+6(z-1)=0$

$x-2+2y-4+6z-6=0$

$x+2y+6z=2+4+6$

$x+2y+6z=12$

normal line:

$r(t) = r_0 + tv$

$= \langle 2,2,1 \rangle + t\langle 1,2,6 \rangle$

$= \langle 2+t, 2+2t, 1+6t \rangle$

$x(t) = 2+t$

$y(t) = 2+2t$

$z(t) = 1+6t$

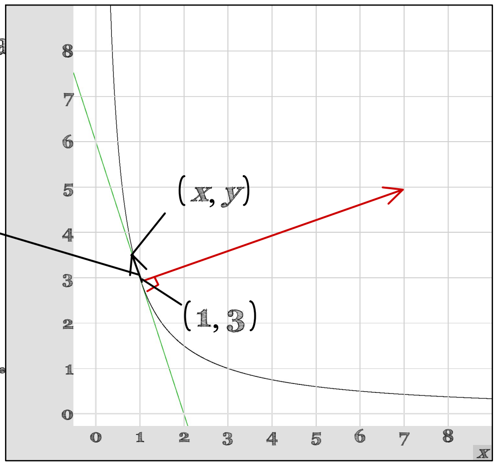
scale $\nabla F(2,2,1)$ by 4 : $\left\langle \frac{4}{4}, \frac{8}{4}, \frac{24}{4} \right\rangle$

$\langle 1,2,6 \rangle$

7. If $f(x,y) = 2xy$, find the gradient vector $\nabla f(1,3)$ and use it to find the tangent line to the level curve f(x,y)=6 at the point (1,3).

$\nabla f(x,y) = \left\langle 2y\frac{\partial}{\partial x}x, 2x\frac{\partial}{\partial y}y \right\rangle = \langle 2y, 2x \rangle$ $\qquad 2xy = 6 \Rightarrow xy = 3 \Rightarrow y = \frac{3}{x}$

The gradient vector is perpendicular to the tangent line, so we ha



$\nabla f(1,3) = \langle 2\cdot 3, 2\cdot 1 \rangle = \langle 6, 2 \rangle$
$\nabla f(1,3) \bullet \langle x-1, y-3 \rangle = 0$

$\langle x-1, y-3 \rangle$

$\langle 6,2 \rangle \bullet \langle x-1, y-3 \rangle = 0$

$6(x-1)+2(y-3)=0$

divide 2 away: $3(x-1)+(y-3)=0$

$3x-3+y-3=0$

$3x+y=3+3$

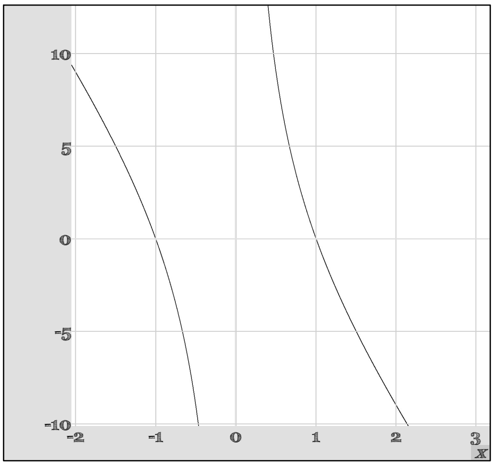
$3x+y=6 \Rightarrow y=6-3x$

From gradient we get that $2/6 = 1/3$ is the slope of the red arrow. So the negative reciprocal is $\frac{-3}{1}$ or the direction vector $\langle 1,-3 \rangle$ Using this we get the equation of the line as $r(t) = \langle 1,3 \rangle + t\langle 1,-3 \rangle = \langle 1+t, 3-3t \rangle$

Notice that at t=0, $r'(t) \bullet \nabla f(1,3) = \langle 1,-3 \rangle \bullet \langle 6,2 \rangle$

$= 6-6 = 0$ so they really are perpendicular.

Notice that in general, we get $r'(t) \bullet \nabla f(x,y) = \langle 1,-3 \rangle \bullet \langle 2y, 2x \rangle$



$= 2y - 6x$

but along level curves, we have $2xy = a$

Thus we get $2\left(\frac{a}{2x}\right) - 6x$ $\qquad y = \frac{a}{2x}$

$\frac{a}{x} - 6x$

With a=6 as above

we get $\frac{6}{x} - 6x$, which is 0 at x=1 and x=-1, so at x=-1, we'd also have a perp. gradient vector .

The plane y+z=3 intersects the cylinder $x^2+y^2=5$ in an ellipse. Find the parametric equations for the tangnet line to this ellipse at the point (1,2,1) and do a graph.

$y+z=3$ is a level surface of the function $f(x,y,z)=y+z$

$x^2+y^2=5$ is a level surface of $g(x,y,z)=x^2+y^2$

$\nabla f(x,y,z)=\langle 0,1,1\rangle$

$\nabla g(x,y,z)=\langle 2x,2y,0\rangle$

Tangent line would be perpendicular to the gradient vectors at $(1,2,1)$

$$v=\nabla f(1,2,1)\times\nabla g(1,2,1)=\begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 2 & 4 & 0 \end{vmatrix}=\langle 0-4,-(0-2),0-2\rangle=\langle -4,2,-2\rangle$$

So the parametric equations of the line are $r(t)=\langle 1,2,1\rangle+t\langle -2,1,-1\rangle$ since $\langle -2,1,-1\rangle=\frac{\langle -4,2,-2\rangle}{2}$