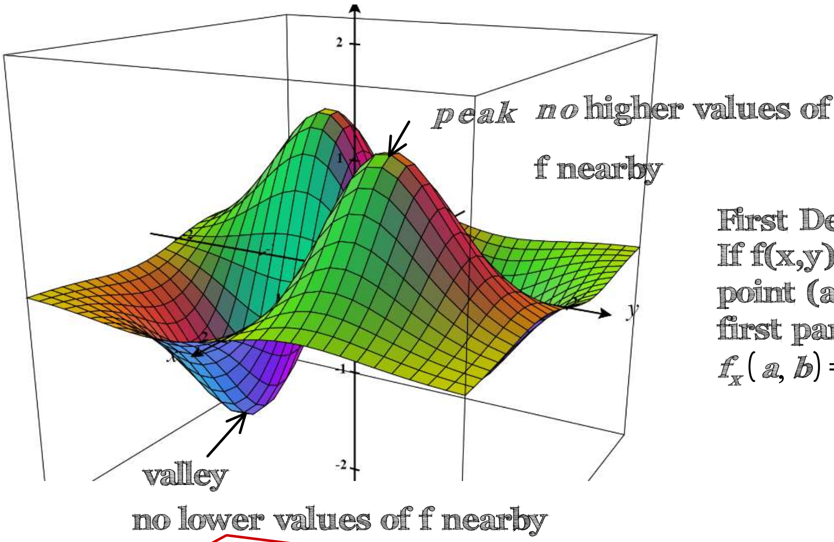
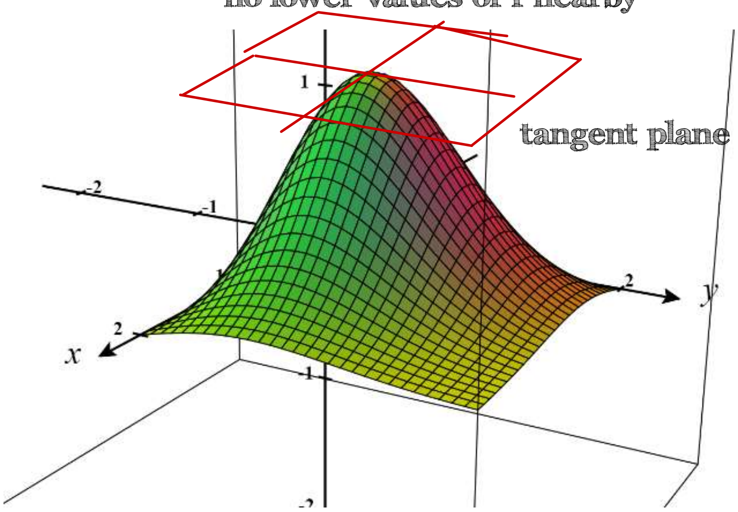
Local Maximum, Local Minimum:
f(a,b) is a local maximum value of f if $f(a,b) \ge f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b). (These are peaks.)
f(a,b) is a local minimum value of f if $f(a,b) \le f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b). (These are valleys.)
When the tangent planes exist at such points, they are horizontal.

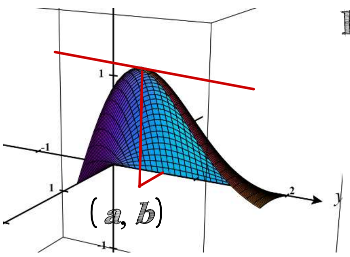


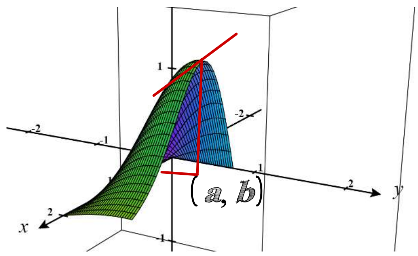
First Derivatives Test for Local Max/Min:
If f(x,y) is a local max or min at an interior point (a,b) of the domain of f, and if the first partial derivatives exist, then
$f_x(a,b) = 0$ and $f_y(a,b) = 0$.



If f has a local extremum at (a,b), then the function $h(y) = f(a,y)$, (x is fixed at a, so y varies)

has $h'(b) = 0$, so $f_y(a,b) = 0$





The function $g(x) = f(x,b)$ has $g'(a) = 0$, so

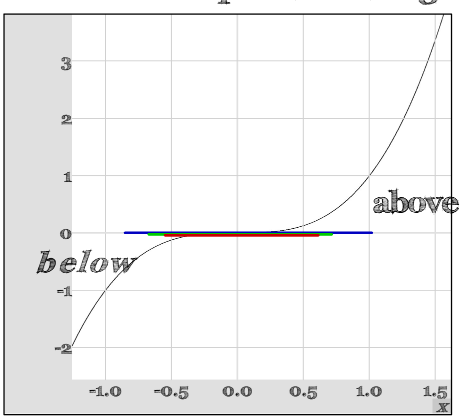
$f_x(a,b) = 0$.

Putting the facts above into the equation of a plane we get $z = f(a,b) + f_x(a,b)(x - x_0) + f_y(a,b)(y - y_0)$

$z = f(a,b) + 0(x - x_0) + 0(y - y_0)$

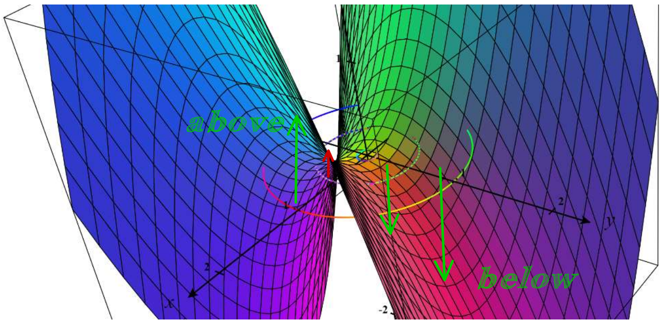
$z = f(a,b)$ (So the tangent plane is horizontal (slopes are zero, so the plane is not tilted)

Critical points are points where $f_x = 0$, $f_y = 0$, or where one or both of the derivatives do not exist.
Just like a function of one variable can have an inflection point, a function of two variables can have a saddle point. In the graph below, every interval gives values of f above and below the x axis.



A differentiable function f(x,y) has a saddle point at a critical point (a,b) if in every open disk centered at (a,b) thre are domain point (a,y) where $f(x,y) > f(a,b)$ and domain points where $f(x,y) < f(a,b)$. The corresponding point (a,b,f(a,b)) on the surface z=f(x,y) is called a saddle point of the surface.

Thus, in the graph below, we see that as we move towards (0,0), every disk has points such that $f>0$ and $f<0$.



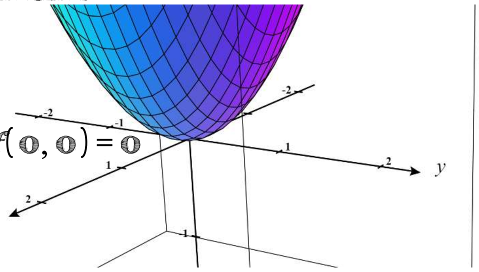
Find the local extreme values of $f(x,y)=x^2+y^2$

$f_x=2x$ $\quad f_y=2y$

$2x=0$ $\quad 2y=0$

$x=0$ $\quad y=0$

So the only critical point is (0,0), where f is $f(0,0)=0^2+0^2=0$. Notice that $x^2+y^2\geq 0$, so z=0 gives a local minimum.



Identifying a saddle point :

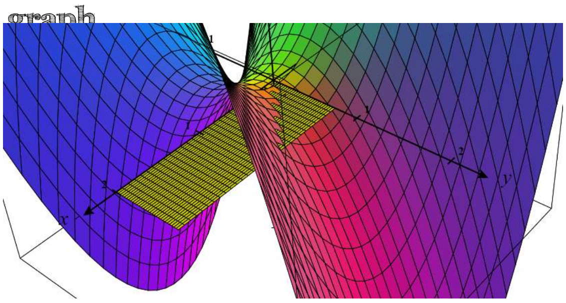
$f(x,y)=x^2-y^2$

$f_x=2x,\ f_y=-2y$

Notice along x=0: $f(0,y)=-y^2<0\ (y\neq 0)$

Notice along y=0: $f(x,0)=x^2>0,\ (x\neq 0)$

Result is that every open disk centered on the origin produces both positive and negative values, so f has no local extreme value. Notice in this case the tangent plane at (0,0) is both above and below the graph.



Second Derivative Test for Extreme Values:
(Proof can be done using Taylor's Formula)
(Look in section 14.10,11th Ed. of Calculus by Thomas)

Suppose that f(x,y) and its first and second partial derivatives are continuous thoroughout a disk centered at (a,b) and that $f_x(a,b)=f_y(a,b)=0$. Then

i. f has a local maximum at (a,b) if $f_{xx}<0$ and $f_{xx}f_{yy}-f_{xy}^2>0$ at (a,b).

ii.f has a local minimum at (a,b) if f_{xx} <0 and f_{xx} f_{yy} -f_{xy}^2 >0 at (a,b).

iii. f has a saddle point at (a,b) if $f_{xx}f_{yy}-f_{xy}^2<0$ at (a,b)

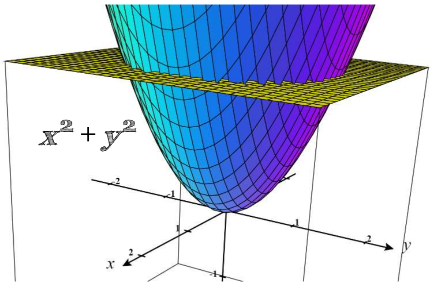
iv. the test is inconclusive at (a,b) if f_{xx} f_{yy} -f_{xy}^2 =0 at(a,b). Must use some other means to determine the behavior of f at (a,b).

The expression $f_{xx}f_{yy}-f_{xy}^2$ is called the discriminant or Hessian of f. It is easier to remember by writing it in determinant form :

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy}-f_{xy}f_{xy} \qquad (f_{xy}=f_{yx})$$

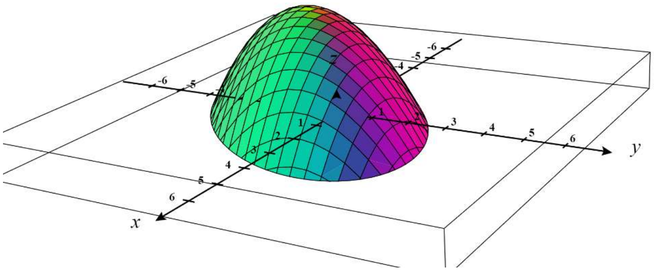
$$=f_{xx}f_{yy}-f_{xy}^2$$

More intuitive interpretations of the statements above:
If the discriminant is positive at the point (a,b), then the surface curves the same way in all directions .
That is , if $f_{xx}<0$, and the discrimiant is positive, there is a local max. If $f_{xx}>0$, and the discriminat is positive, there is a local min. This is shown below for x^2+y^2 where $f_{xx}=2>0$.



$f_{xx}=2>0$

Determine the local minima and maxima of the function $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$.



Notice there is a single hump in the graph.

$$f_x = \frac{\partial}{\partial x}\left[xy - x^2 - y^2 - 2x - 2y + 4\right] = y - 2x - 2$$

$$f_y = \frac{\partial}{\partial y}\left[xy - x^2 - y^2 - 2x - 2y + 4\right] = x - 2y - 2$$

$$f_x = 0 \Rightarrow y - 2x - 2 = 0 \Rightarrow y = 2x + 2$$

$$f_y = 0 \Rightarrow x - 2y - 2 = 0 \Rightarrow x - 2(2x + 2) - 2 = 0$$

$$\Rightarrow x - 4x - 4 - 2 = 0$$

$$\Rightarrow -3x - 6 = 0$$

$$\Rightarrow -3x = 6$$

$$\Rightarrow x = -2$$

Now find y:

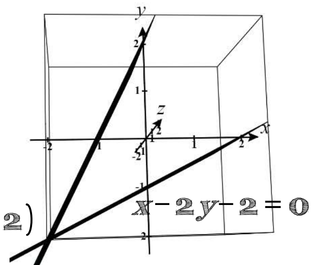
$$x = -2: \ -2 - 2(y) - 2 = 0$$

$$-4 - 2y = 0$$

$$-2y = 4$$

$$y = -2$$

$y = 2x + 2$



$(-2, -2)$

$x - 2y - 2 = 0$

Calculate the ingredients for the 2nd partials test:

$$f_{xx} = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}(y - 2x - 2) = -2$$

$$f_{yy} = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}(x - 2y - 2) = -2$$

$$f_{yx} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}(x - 2y - 2) = 1$$

Plug into the second partials formulas:

i. f has a local maximum at (a,b) if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b).

ii. f has a local minimum at (a,b) if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b).

iii. f has a saddle point at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a,b)

iv. the test is inconclusive at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 = 0$ at(a,b). Must use some other means to determine the behavior of f at (a,b).

Here, $f_{xx} = -2 < 0$

$$D(-2,-2) = (-2)(-2) - (1)^2 = 4 - 1 = 3 > 0$$

So we have the case of a local maximum at (-2,-2), where the value of the function is

$$f(-2,-2) = (-2)(-2) - (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4$$

$$= 8$$

Searching for Local Extreme Values:

Find the local extreme values of $f(x,y) = 2xy$

Notice f is diffenerntiable everywhere, so it can assume extreme values only where

$$f_x = \frac{\partial}{\partial x}(2xy) = 2y \Rightarrow 2y = 0 \Rightarrow y = 0$$

$$f_y = \frac{\partial}{\partial y}(2xy) = 2x \Rightarrow 2x = 0 \Rightarrow x = 0$$

So only the point is a candidate for an extreme value.

$$f_{xx} = \frac{\partial}{\partial x} 2y = 0, \ f_{yy} = \frac{\partial}{\partial y} 2x = 0, \ f_{xy} = \frac{\partial}{\partial y} 2y = 2$$

So applying the second partials test we get

i. f has a local maximum at (a,b) if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b).

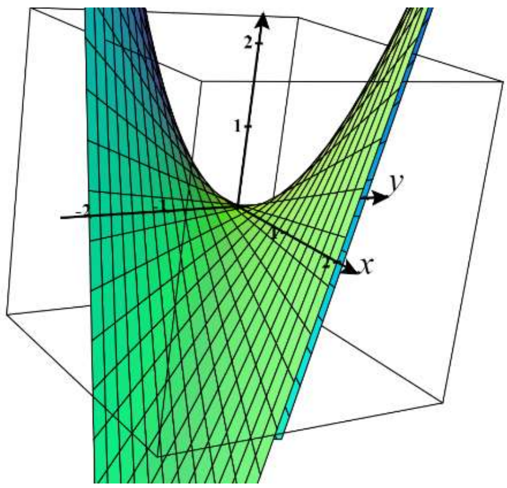
ii. f has a local minimum at (a,b) if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b).

iii. f has a saddle point at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a,b)

iv. the test is inconclusive at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 = 0$ at(a,b). Must use some other means to determine the behavior of f at (a,b).

$f_{xx} f_{xx} - f_{xy}^2 = 0 \cdot 0 - (2)^2 = -4 < 0$, so we have a saddle point at (0,0).

The graph of f reinforces the result above .



Absolute Maxima and Minima on Closed, Bounded Regions:

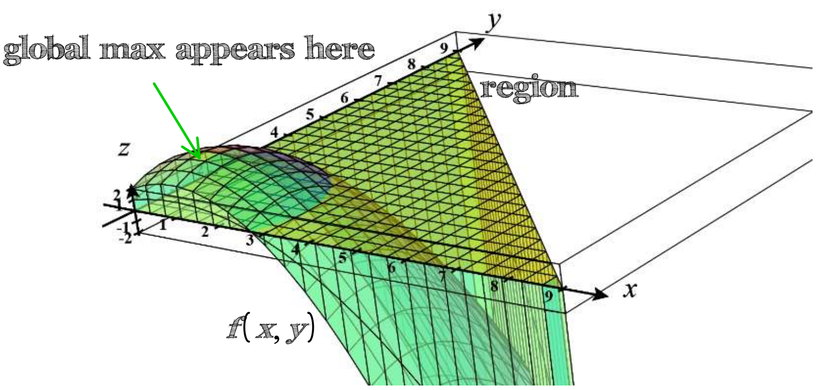
1. List the interior points of R where f may have local maxima or minima and evaluate f at these points. These are the critical points of f.

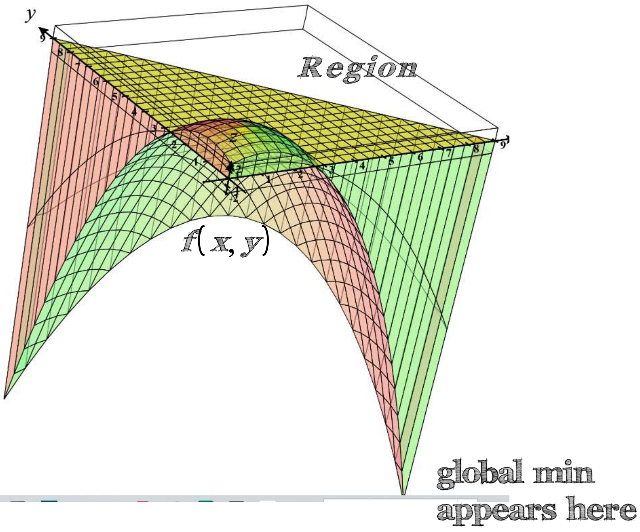
2. List the boundary points of R where f has local maxima and minima and evaluate f at these points.

Look through the lists above and pick out the max or min values.

Finding absolute extrema.

Maximize $f(x,y)=2+2x+2y-x^2-y^2$ over the triangle given by (0,0),(9,0) and (0,9)





Interior points:

$\nabla f(x,y)=\langle 0,0\rangle$

$f_x=\frac{\partial}{\partial x}(2+2x+2y-x^2-y^2)=2-2x \Rightarrow 2-2x=0 \Rightarrow -2x=-2 \Rightarrow x=1$

$f_y=\frac{\partial}{\partial y}(2+2x+2y-x^2-y^2)=2-2y \Rightarrow 2-2y=0 \Rightarrow -2y=-2 \Rightarrow y=1$

At (1,1), we get $f(1,1)=2+2\cdot 1+2\cdot 1-1^2-1^2=2+2+2-1-1=4$

Boundary Considerations:

along $y=0$, we get $f(x,0)=2+2x+2\cdot 0-x^2-0^2=2+2x-x^2,\ 0\le x\le 9$

Points/values

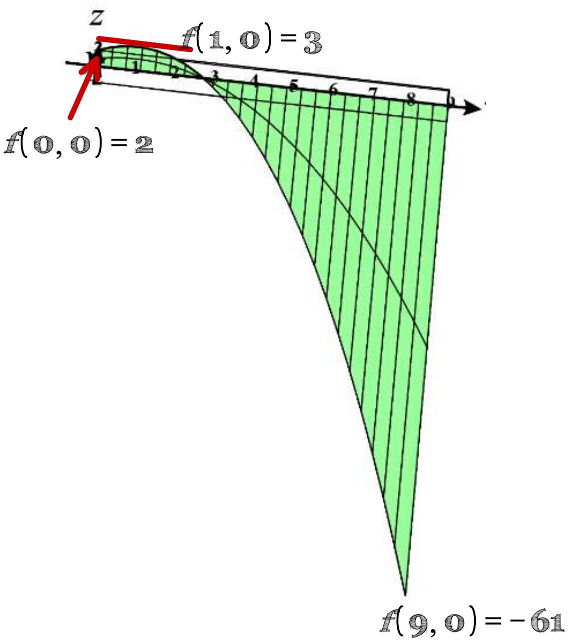
$f(1,1)=4$

$f(9,0)=-61$

$f(1,0)=3$

$f(0,0)=2$

$f(4.5,4.5)=-20.5$



This is an inverted parabola and we can tell the minimum is at x=9, where we get

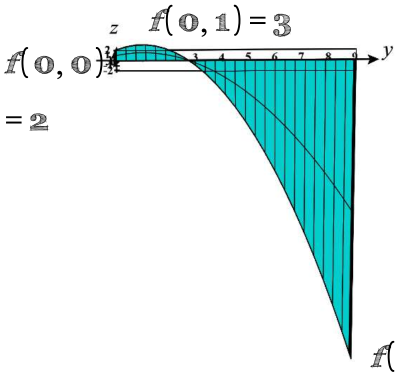
$f(9,0)=2+2(9)-(9)^2=2+18-81=-61$

Also,notice that $f'(x,0)=2-2x=0 \Rightarrow -2x=-2 \Rightarrow x=1$

where f(1,0)=2+2(1)-1² =4-1=3

When x=0, we get $f(0,y)=2+2(0)+2y-0^2-y^2=2+2y-y^2,\ 0\le y\le 9$

This is shown in the graph below.



$f(0,9)=-61$

Since here only the variable is different, but the expression is the same as bove, we can also say that

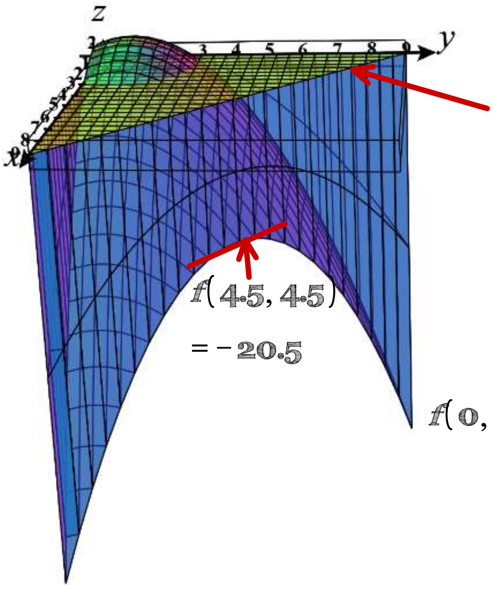
f(0,0)=2

f'(0,y)= $2-2y \Rightarrow 2-2y=0 \Rightarrow y=1$ where $f(0,1)=2+2\cdot 1-1^2=3$

$f(0,9)=-61$

These values are already listed above where it says points/values.

Lastly, we have to look at the values of f along line $y = 9 - x$



$y = 9 - x$

Along this line, we get $f(x, 9-x) = 2 + 2x + 2(9-x) - x^2 - (9-x)^2$

$f(9,0) = -61$

$f(0,9) = -61$

$$= 2 + 2x + 18 - 2x - x^2 - (9-x)^2$$

$$= 20 - x^2 - (9-x)^2$$

$f(0,9) = -61$

$$f'(x, 9-x) = -2x - 2(9-x)(-1)$$

$$= -2x + 2(9-x)$$

$$= -2x + 18 - 2x$$

$$= -4x + 18$$

$f(9,0) = -61$

$f'(x, 9-x) = 0$

$-4x + 18 = 0$

$x = \frac{18}{4} = 4.5$

Points/values

$f(1,1) = 4$

$f(9,0) = -61$

$f(1,0) = 3$

$f(0,0) = 2$

$f(4.5, 4.5) = -20.5$

When x=4.5, y=9-4.5=4.5 also.

So we get

$f(4.5, 4.5) = 20 - 4.5^2 - (9-4.5)^2 = -20.50$

So there max value is 4 at (1,1) and the minimum value is -61 at (0,9) and (9,0).