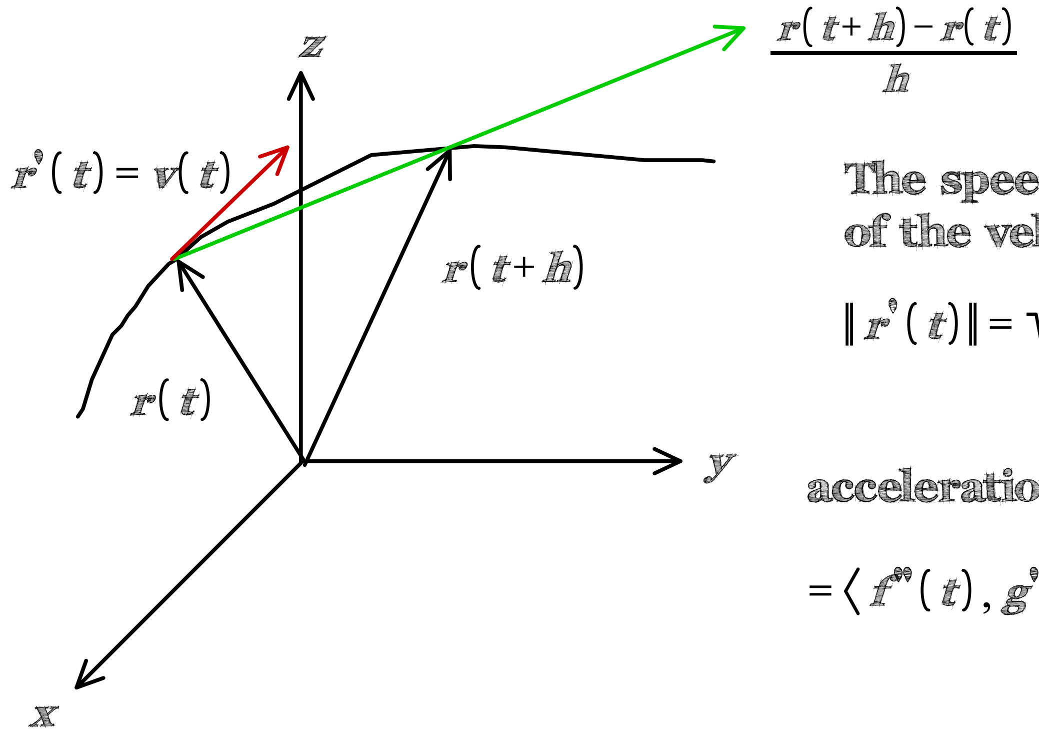
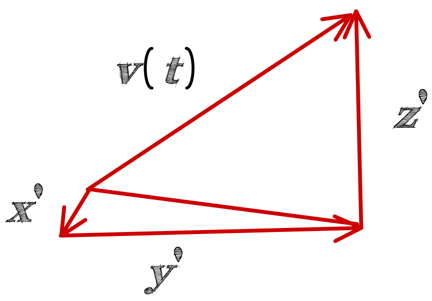
Velocity and Acceleration:

rate of change of distance with respect to time$=\lim_{h\to 0}\frac{r(t+h)-r(t)}{h}$ where $r=\langle f(t), g(t), h(t)\rangle$



The speed of the particle is $\|r'(t)\|=\|v(t)\|$. It's the magnitude of the velocity vector.

$$\|r'(t)\|=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}$$



$$\text{acceleration}=\frac{d}{dt}r'(t)=r''(t)$$

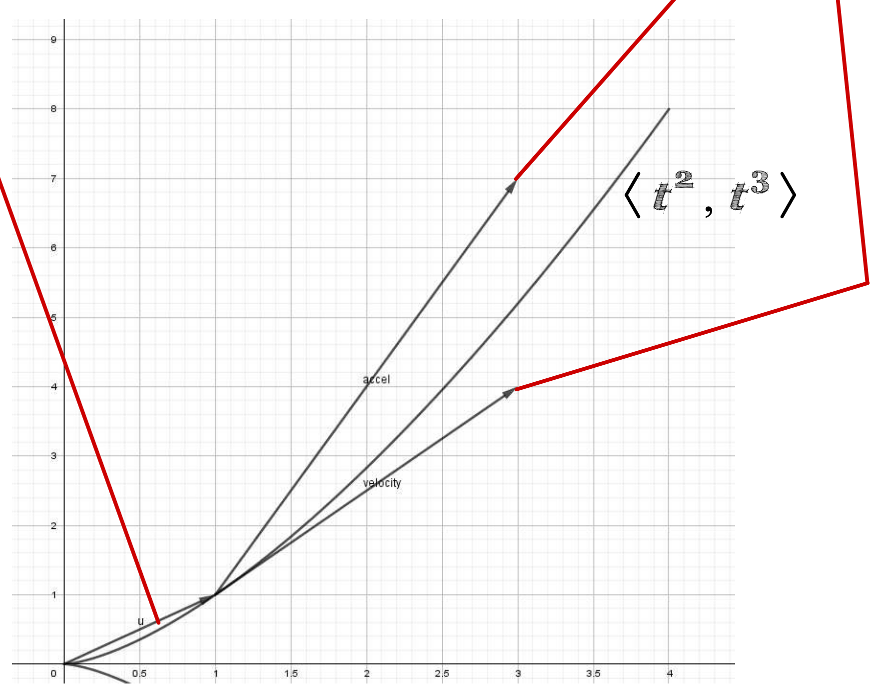
$$=\langle f''(t), g''(t), h''(t)\rangle$$

Example: The position of an object is given by $r(t)=\langle t^2, t^3\rangle$. Find the velocity, speed and acceleration when t=1 and draw a picture.

$r'(t)=\left\langle \frac{d}{dt}t^2, \frac{d}{dt}t^3\right\rangle=\langle 2t, 3t^2\rangle \quad \Rightarrow r'(1)=\langle 2\cdot 1, 3\cdot 1^2\rangle=\langle 2,3\rangle \quad \|r'(1)\|=\sqrt{2^2+3^2}=\sqrt{13}\approx 3.61$ (speed)

$r''(t)=\left\langle \frac{d}{dt}2t, \frac{d}{dt}3t^2\right\rangle=\langle 2, 6t\rangle \quad \Rightarrow r''(1)=\langle 2, 6\cdot 1\rangle=\langle 2,6\rangle$

$r(1)=\langle 1^2, 1^3\rangle=\langle 1,1\rangle$



Find the velocity, acceleration and speed of a particle with position vector r(t)$=\langle t, e^t, te^t\rangle$

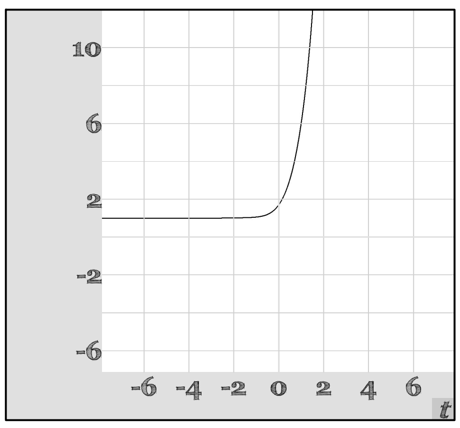
$$r'(t)=\frac{d}{dt}r(t)=\left\langle \frac{d}{dt}t, \frac{d}{dt}e^t, \frac{d}{dt}te^t\right\rangle=\langle 1, e^t, e^t+te^t\rangle=\langle 1, e^t, e^t[1+t]\rangle$$

$$\text{speed}=\|r'(t)\|=\sqrt{1^2+(e^t)^2+[e^t(1+t)]^2}=\sqrt{1+e^{2t}+[e^{2t}(1+t)^2]}$$

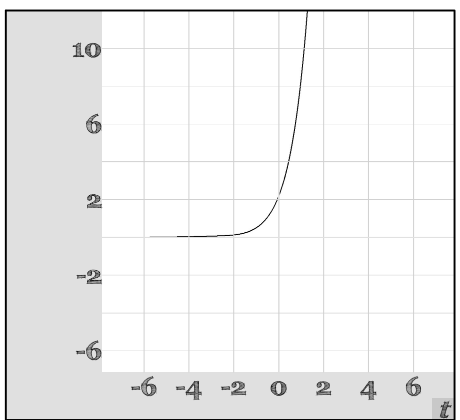
$$a(t)=\frac{d}{dt}r'(t)=r''(t)=\left\langle \frac{d}{dt}1, \frac{d}{dt}e^t, \frac{d}{dt}e^t[1+t]\right\rangle=\langle 0, e^t, e^t+e^t[1+t]\rangle=\langle 0, e^t, e^t[1+1+t]\rangle$$

$$=\langle 0, e^t, e^t[2+t]\rangle$$

$$\|a(t)\|=\sqrt{e^{2t}+(e^t[2+t])^2}$$



speed vs. time



acceleration vs. time

A moving particle starts at an initial position r(0)= $\langle 2, 0, 0 \rangle$ with initial velocity $v(0) = \langle 0, -1, 1 \rangle$. Its acceleration is a(t)= $\langle 2t, 3t, 5 \rangle$. Find its velocity and position at time t.

v(t)= $\int a(t)\,dt = \int \langle 2t, 3t, 5 \rangle\,dt = \left\langle t^2, \frac{3}{2} t^2, 5t \right\rangle + C$

Since $v(0) = \langle 0, -1, 1 \rangle$, we get $\left\langle 0^2, \frac{3}{2} 0^2, 5 \cdot 0 \right\rangle + C = \langle 0, -1, 1 \rangle$

$$\langle 0, 0, 0 \rangle + C = \langle 0, -1, 1 \rangle$$
$$C = \langle 0, -1, 1 \rangle - \langle 0, 0, 0 \rangle$$
$$C = \langle 0, -1, 1 \rangle$$

So the complete velocity function is $v(t) = \left\langle t^2, \frac{3}{2} t^2 - 1, 5t + 1 \right\rangle$

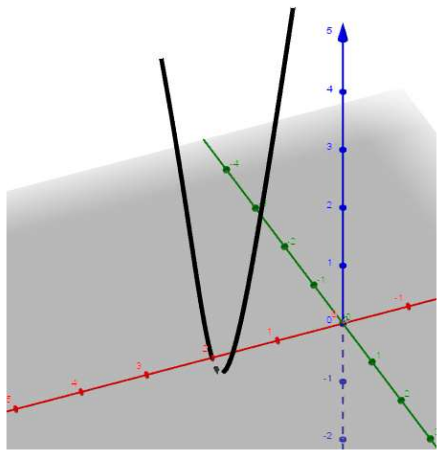
$r(t) = \int v(t)\,dt = \int \left\langle t^2, \frac{3}{2} t^2 - 1, 5t + 1 \right\rangle dt + D$

$= \left\langle \frac{t^3}{3}, \frac{1}{2} t^3 - t, \frac{5}{2} t^2 + t \right\rangle + D$

Since $r(0) = \langle 2, 0, 0 \rangle$, we get $\left\langle \frac{0^3}{3}, \frac{1}{2} 0^3 - 0, \frac{5}{2} 0^2 + 0 \right\rangle + D = \langle 2, 0, 0 \rangle$

$$\langle 0, 0, 0 \rangle + D = \langle 2, 0, 0 \rangle$$
$$D = \langle 2, 0, 0 \rangle - \langle 0, 0, 0 \rangle$$
$$D = \langle 2, 0, 0 \rangle$$

So the complete position function is $r(t) = \left\langle \frac{t^3}{3}, \frac{1}{2} t^3 - t, \frac{5}{2} t^2 + t \right\rangle + \langle 2, 0, 0 \rangle$



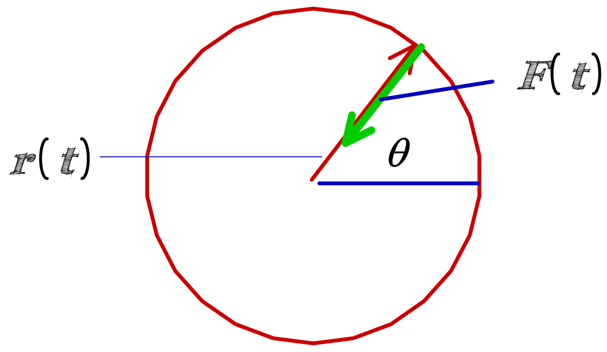
$= \left\langle \frac{t^3}{3} + 2, \frac{1}{2} t^3 - t, \frac{5}{2} t^2 + t \right\rangle$

Newton's Law of Motion:

$F(t) = m \cdot a(t)$, F= force, a=acceleration

In the case of circular motion, r(t)= $\langle a\cos(\omega t), a\sin(\omega t) \rangle$.

period $= \frac{2\pi}{\omega}$ (time to complete a cycle)



$a(t) = \frac{d}{dt} \frac{d}{dt} \langle a\cos(\omega t), a\sin(\omega t) \rangle = \frac{d}{dt} \langle -a\sin(\omega t)\omega, a\cos(\omega t)\omega \rangle$

$$= \frac{d}{dt} \langle -a\omega \sin(\omega t), a\omega \cos(\omega t) \rangle$$
$$= \langle -a\omega \cos(\omega t)\omega, -a\omega \sin(\omega t)\omega \rangle$$
$$= \langle -a\omega^2 \cos(\omega t), -a\omega^2 \sin(\omega t) \rangle$$
$$= -\omega^2 \langle a\cos(\omega t), a\sin(\omega t) \rangle$$
$$= -\omega^2 r(t)$$

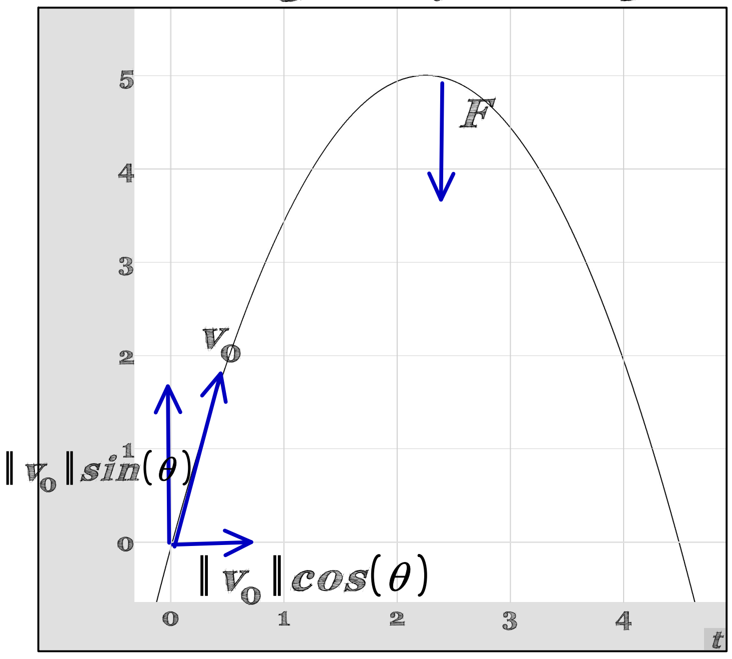
Thus, the force is given by $F(t) = m[-\omega^2 r(t)] = -m\omega^2 r(t)$ So the force is a vector that points inward (b/c of the negative) and is a scalar multiple of the position vector for each t .

Here, both the mass, m and the the angular speed, $\omega(omega)$, have an impact.The defintion of omega is $\frac{d\theta}{dt} = \omega$ (rate at which the angle changes with respect to time)

Projectile Motion:

In the absense of air resistance, an object moves only under the force of gravity.

Acceleration near the surface of the Earth is given by g=-9.81m/s^2. So the force acting on an object mass m is given by $F = -mgj$, where j=$\langle 0, 1 \rangle$ (F acts downward,hence the -).



From the graph, we see that the initial velocity can be written as $v_0 = \langle \|v_0\| cos(\theta), \|v_0\| sin(\theta) \rangle$

Since a(t)=-gj (negative g since it's downward)

, we can integrate to get $v(t) = \int -gdt\ j = -gt\,j + v_0$

Integrting v(t), we get

$r(t) = \int v(t)\,dt = \int -gtdt\ j + \int v_0\,dt = -\frac{g}{2} t^2 j + v_0 t + D$

Remember form above that $v_0 = \langle \|v_0\| cos(\theta), \|v_0\| sin(\theta) \rangle$,

so we get $r(t) = \frac{-g}{2} t^2 j + \langle \|v_0\| cos(\theta), \|v_0\| sin(\theta) \rangle t + D$

$$r(t) = \left\langle 0, \frac{-g}{2} t^2 \right\rangle + \langle \|v_0\| cos(\theta) t, \|v_0\| sin(\theta) t \rangle + D$$

$$r(t) = \left\langle \|v_0\| cos(\theta) t, \frac{-g}{2} t^2 + \|v_0\| sin(\theta) t \right\rangle + D$$

If we assume that the object is launched from (0,y_0), then D = $\langle 0, y_0 \rangle$

From r(t), we get the parametric equations of the motion of the object as

$x(t) = \|v_0\| cos(\theta) t,\ y(t) = \frac{-g}{2} t^2 + \|v_0\| sin(\theta) t + y_0$

How far does the object go down field? That occurs when y=0(striking the ground)

So $y(t) = 0$

$\frac{-g}{2} t^2 + \|v_0\| sin(\theta) t = 0$

$t \left[\frac{-g}{2} t + \|v_0\| sin(\theta) \right] = 0$

$t = 0$ (when it's first launched) $\quad \frac{-g}{2} t + \|v_0\| sin(\theta) = 0 \rightarrow t = \frac{2\|v_0\| sin(\theta)}{g}$ (when it strikes the ground)

Thus we get $x\left(\frac{2\|v_0\| sin(\theta)}{g} \right) = \|v_0\| cos(\theta) \cdot \frac{2\|v_0\| sin(\theta)}{g} = \frac{\|v_0\|^2 \cdot 2 cos(\theta) sin(\theta)}{g} = \frac{\|v_0\|^2 sin(2\theta)}{g}$

set $\|v_0\|^2 = 1$ (this is so we can focus on sin(2 θ)]

$g = 9.81$

This is maximized when $\theta = 45°$.

Thus, if a proctile is fired so it leaves the barrel at a speed of 200m/s (meters per second) and from height of 2 meters, at an angle of 30 degrees, we get the equations

$x(t) = 200\, cos(30) t,\ y(t) = \frac{-9.81}{2} t^2 + 200\, sin(30) t + 2$

$= 200 \frac{\sqrt{3}}{2} t \qquad = -4.9 t^2 + 200 \cdot \frac{1}{2} t + 2$

$= 100 \sqrt{3}\ t \qquad = -4.9 t^2 + 100 t + 2$

By the Quadratic Formula, we get

$t = \frac{-100 \pm \sqrt{100^2 - 4(-4.9) \cdot 2}}{2(-4.9)}$

$t = \frac{100 \mp 100.196}{9.8}$

$t = \frac{100 + 100.196}{9.8} \approx 20.428$

So it takes the projectile 20.428 seconds of time to hit the ground.

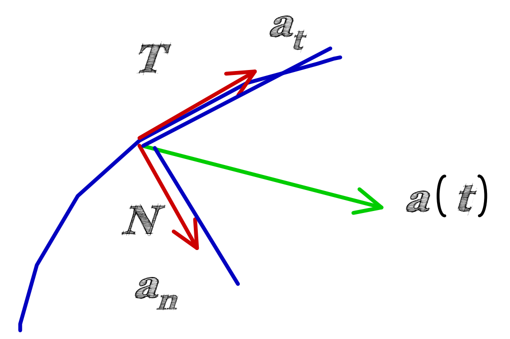
Thus, the distance in meters is

$x(20.428) = 100\sqrt{3} \cdot \frac{100 + 100.196}{9.8} \approx 3538.262$ meters

The velocity is v(t)= $\langle 100\sqrt{3}, -9.81t+100\rangle$ (just differentiate r(t) part by part)

The speed on impact is $\|v(20.428)\| = \sqrt{(100\sqrt{3})^2+(-0.81(20.428)+100)^2} \approx 192.26143\ m/s$

Acceleration can be resolved into parallel and perpendicular componets.



Remember that we can get a unit tangent vector by doing

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

From this, we get ||r'(t)||T(t)=r'(t)

$$a = \frac{d}{dt}r'(t) = \frac{d}{dt}\|r'(t)\|T(t) = \|r'(t)\|' T(t) + \|r'(t)\| T'(t)$$

$\left(\text{curvature, not developed here is given by } \kappa = \frac{\|T'(t)\|}{\|r'(t)\|} \Rightarrow \|T'(t)\| = \|r'(t)\|\kappa\right)$

$\left(\text{the unit normal vector is } N(t) = \frac{T'(t)}{\|T'(t)\|} \Rightarrow \|T'(t)\|N(t) = T'(t)\right)$

Thus we get $a = \|r'(t)\|' T(t) + \|r'(t)\| \|T'(t)\| N(t)$

$$a = \|r'(t)\|' T(t) + \|r'(t)\| \|r'(t)\| \kappa N(t)$$

writing $v = \|r'(t)\|$, we get $a = v' T(t) + v\, v \kappa N(t)$

$$a = v' T(t) + v^2 \kappa N(t)$$

So we get the normal component of acceleration as $v^2\kappa$ and we get the tangential component of acceleration as v'.

$a_n = \kappa v^2,\ a_t = v'$

Since we're often given r(t), it would be helpful to get expressions for a_n and a_t in terms of r,r' anc

Notice that

(speed· direction=vT)

a= $v' T(t) + v^2 \kappa N(t)$ from above

$\vec{v} \bullet a = vT \bullet (v'T + v^2\kappa N) = vv'T \bullet T + v^3 \kappa T \bullet N$ (Remember that $T \bullet N = 0$, they're $\perp$, $T \cdot T = 1$)

$$= vv'$$

So now solve for $v' = a_t = \frac{\vec{v} \bullet \vec{a}}{v} = \frac{r'(t) \bullet r''(t)}{\|r'(t)\|}$ (remember that $v' = a_t$ from above)

Also, $a_n = \kappa v^2 = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}\|r'(t)\|^2 = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|}$ curvature= $\kappa = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$ (reason not shown here)

A particle moves with position given by r(t)= $\langle t, t^2, t^3\rangle$ Find the normal and tangnential componen of acceleration:

$$a_t = \frac{r'(t) \bullet r''(t)}{\|r'(t)\|} = \frac{\langle 1, 2t, 3t^2\rangle \bullet \langle 0, 2, 6t\rangle}{\|\langle 1, 2t, 3t^2\rangle\|} = \frac{1\cdot 0 + 2t(2) + 3t^2(6t)}{\sqrt{1^2+(2t)^2+(3t^2)^2}} = \frac{4t+18t^3}{\sqrt{1+4t^2+9t^4}}$$ (function of t only)

wolfram alpha code

```
[d/dt<t,t^2,t^3> . d/dt d/dt <t,t^2,t^3>]/sqrt(1^2+(2t)^2+(3t^2)^2)
```

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \langle 2t\cdot 6t - 2\cdot 3t^2, -(1\cdot 6t - 0\cdot 3t^2), 1\cdot 2 - 0\cdot 2t\rangle$$

$$= \langle 12t^2 - 6t^2, -(6t), 2\rangle = \langle 6t^2, -6t, 2\rangle$$

So $a_n = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|} = \frac{\|\langle 6t^2, -6t, 2\rangle\|}{\|\langle 1, 2t, 3t^2\rangle\|} = \frac{\sqrt{(6t^2)^2+(-6t)^2+2^2}}{\sqrt{1^2+(2t)^2+(3t^2)^2}} = \frac{\sqrt{36t^4+36t^2+4}}{\sqrt{1+4t^2+9t^4}}$ (function of t only)

```
||{2t,2t,3t^2}*{2,2,6t}||/||{2t,2t,3t^2}||
```

how to type into wolfram alpha