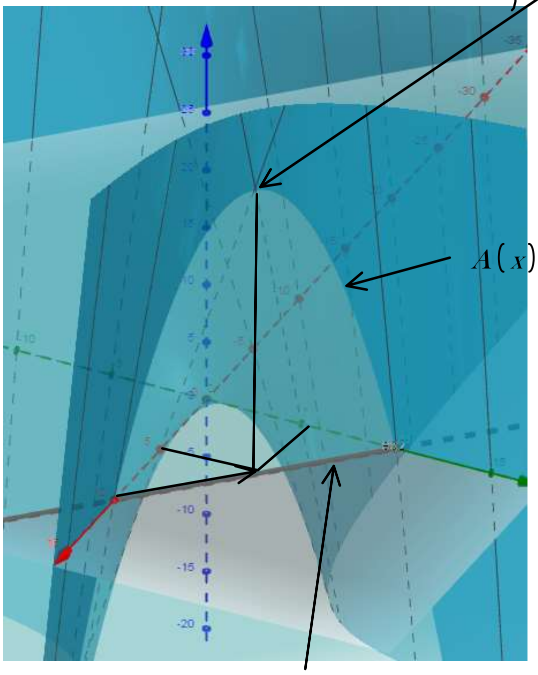
Constrained Optimization

LaGrange Multipliers.

In many application , we must find the extrema of a function f(x,y) subject to a constraint, where a constraint is a curve of the form g(x,y)=k.

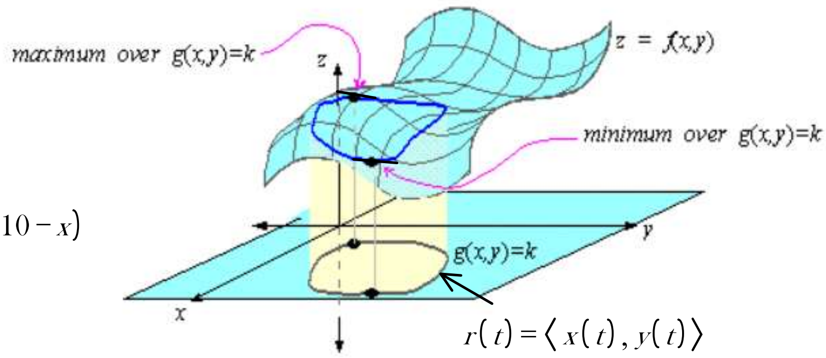
Perimeter is 20: 2x+2y=20 $\rightarrow x+y=10$
Area is $A=xy \Rightarrow A(x)=x(10-x)$

In this case, we see that when x=5, we get $A(x=5)=5(10-5)=25$ max area.



$A(x)=x(10-x)$

constraint curve
$x+y=10$



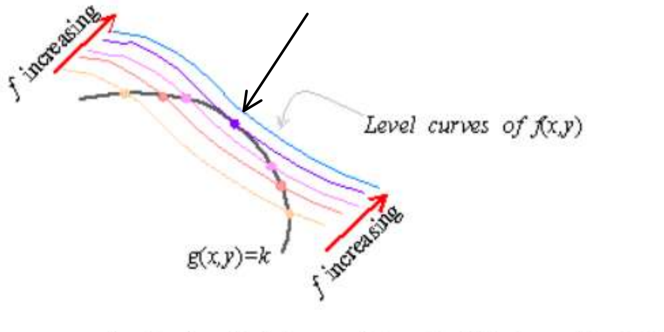
$r(t)=\langle x(t), y(t)\rangle$
for all t [a,b]

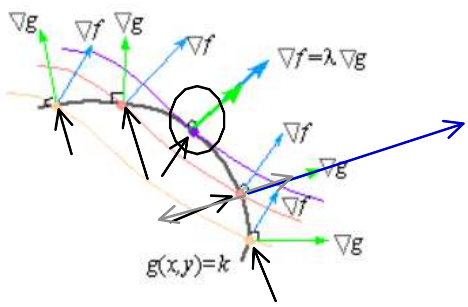
Finding the extrema (max or min) of f(x,y) subject to g(x,y)=k is equivalent to finding the absolute extrema for $z(t)=f(x(t),y(t))$

$$\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=\langle f_x, f_y\rangle \bullet \langle x', y'\rangle = \nabla f \bullet r' = 0$$

That is, the critical points of z(t) ocurs when $\nabla f \perp r'$.

Since $g(x,y)=k$, (kth level curve), then $\nabla g \perp r'$.





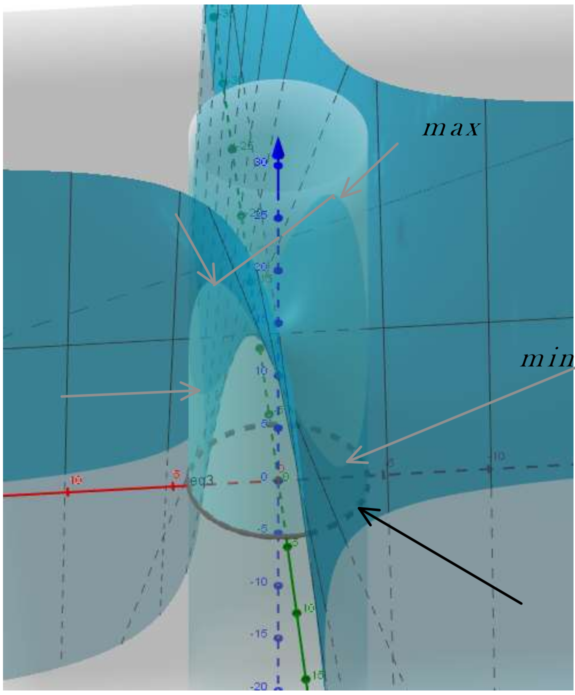
$\nabla f=\lambda \nabla g$, letter we use is called lambda
λ is called the LaGrange Multiplier. Look up Gilbert Stran's Book . Free on online on the MIT Open CourseWare page. Used in economics a lot.

We end up with the following then:
To find the extreme of f(x,y) over g(x,y)=k, we solve

$\nabla f=\lambda \nabla g$
$f_x=\lambda g_x$
$f_y=\lambda g_y$
g(x,y)=k

Find the extrema of f(x,y)=xy+14 subject to the condition $x^2 + y^2 = 18$.

Remembe the curve $x^2 + y^2 = 18$ is the same as the 0th level curve of $g(x, y) = x^2 + y^2 - 18$.



max

min

$$g(x, y) = x^2 + y^2 - 18 = 0$$
$$f(x, y) = xy + 14$$
$$\nabla f(x, y) = \langle y, x \rangle$$
$$\nabla g(x, y) = \langle 2x, 2y \rangle$$
$$y = \lambda 2x$$
$$x = \lambda 2y$$
$$\frac{y}{x} = \frac{\lambda 2x}{\lambda 2y} \Rightarrow \frac{y}{x} = \frac{x}{y} \Rightarrow y^2 = x^2$$
$$x^2 + x^2 = 18$$
$$2x^2 = 18$$
$$x^2 = 9 \Rightarrow x = \pm 3$$
$$y^2 = (\pm 3)^2 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$$
$$(3, 3), (-3, -3), (-3, 3), (3, -3)$$
$$f(3, 3) = 3 \cdot 3 + 14 = 9 + 14 = 23$$
$f(-3, -3) = (-3)(-3) + 14 = 9 + 14 = 23$ max value

$f(3, -3) = 3(-3) + 14 = -9 + 14 = 5$ min value

EXAMPLE 3 Find the greatest and smallest values that the function

$$f(x, y) = xy$$

takes on the ellipse (Fig. 12.62)

$$\frac{x^2}{8} + \frac{y^2}{2} = 1. \qquad g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$$

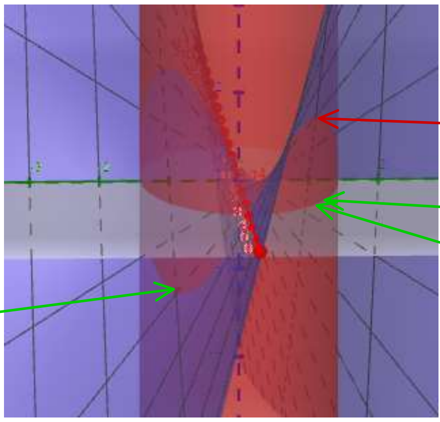
$$f(x, y) = xy$$
$$\nabla f(x, y) = \langle y, x \rangle, \nabla g(x, y) = \left\langle \frac{1}{4} x, y \right\rangle$$
$$\nabla f = \lambda \nabla g$$

$$\left.\begin{aligned} y &= \lambda \frac{1}{4} x \\ x &= \lambda y \\ \frac{x^2}{8} + \frac{y^2}{2} &= 1 \end{aligned}\right\}$$

$$\frac{y}{x} = \frac{0.25x}{y} \Rightarrow y^2 = 0.25x^2$$
$$\frac{x^2}{8} + \frac{(1/4x^2)}{2} = 1$$
$$\frac{x^2}{8} + \frac{1}{8}x^2 = 1$$
$$\frac{2}{8}x^2 = 1$$
$$x^2 = 4 \Rightarrow x = \pm 2$$

$$y^2 = 0.25(\pm 2)^2$$
$$y^2 = \frac{1}{4} \cdot 4$$
$$y^2 = 1 \Rightarrow y = \pm 1$$
$$(2, 1), (-2, 1), (2, -1), (-2, -1)$$
$$f(2, 1) = 2 \cdot 1 = 2, f(-2, 1) = -2 \cdot 1 = -2$$
$$f(2, -1) = 2(-1) = -2$$

$max = 2, min = -2$



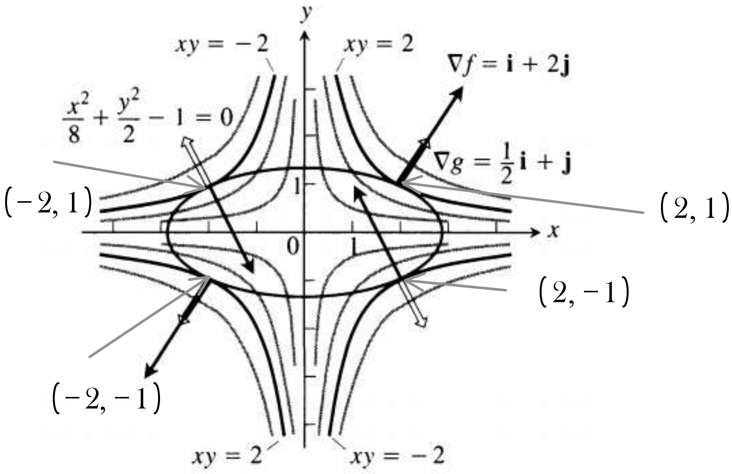
max

$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$

$$f(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$$

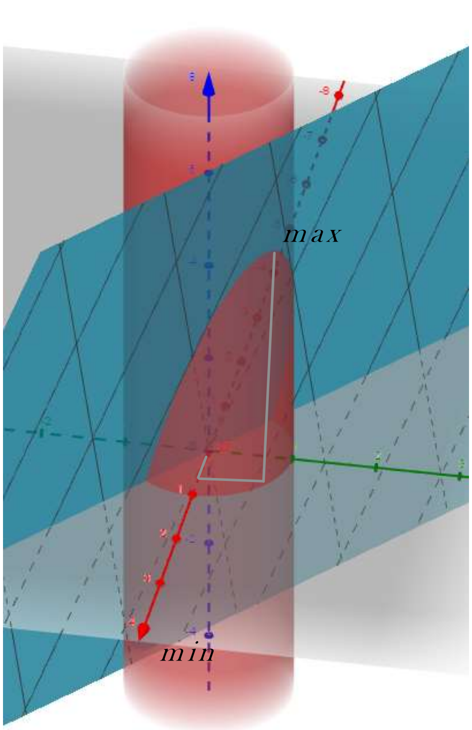
min

$$\left(\frac{x}{\sqrt{8}}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 = 1$$
$$cos(t) = \frac{x}{\sqrt{8}}, sin(t) = \frac{y}{\sqrt{2}}$$
$$x = \sqrt{8}\, cos(t), y = \sqrt{2}\, sin(t)$$
$$f(x(t), y(t)) = z(t) = \sqrt{8}\sqrt{2}\, sin(t) cos(t)$$



Find the max and min values of the function f(x,y)=3x+4y subject to the constraint $x^2 + y^2 = 1$

$$f(x,y) = 3x + 4y \Rightarrow \nabla f(x,y) = \langle 3, 4 \rangle,\ g(x,y) = x^2 + y^2 - 1 = 0 \Rightarrow \nabla g(x,y) = \langle 2x, 2y \rangle$$



$$3 = \lambda 2x \Rightarrow \lambda = \frac{3}{2x}$$

$$4 = \lambda 2y \Rightarrow \lambda = \frac{4}{2y} = \frac{2}{y}$$

$$\frac{3}{2x} = \frac{2}{y} \Rightarrow 3y = 4x \Rightarrow y = \frac{4x}{3}$$

$$x^2 + \left(\frac{4x}{3}\right)^2 = 1 \qquad y = \frac{4}{3}\left(\frac{3}{5}\right) = \frac{4}{5}$$

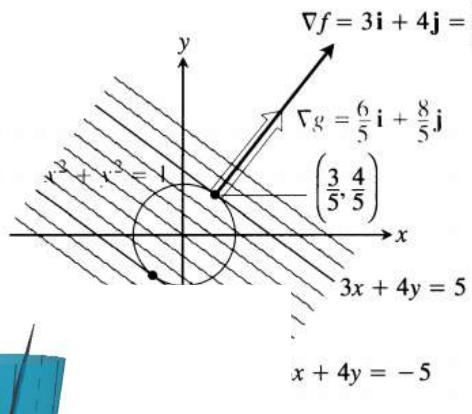
$$x^2 + \frac{16x^2}{9} = 1 \qquad y = \frac{4}{3}\left(-\frac{3}{5}\right) = \frac{-4}{5}$$

$$9x^2 + 16x^2 = 9 \qquad (3/5, 4/5), (-3/5, -4/5)$$

$$25x^2 = 9$$

$$x^2 = \frac{9}{25}$$

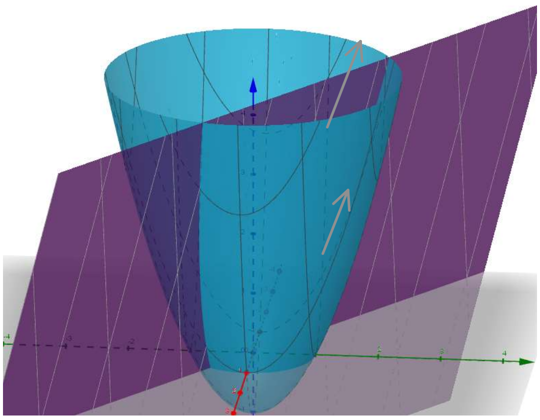
$$x = \pm \frac{3}{5}$$

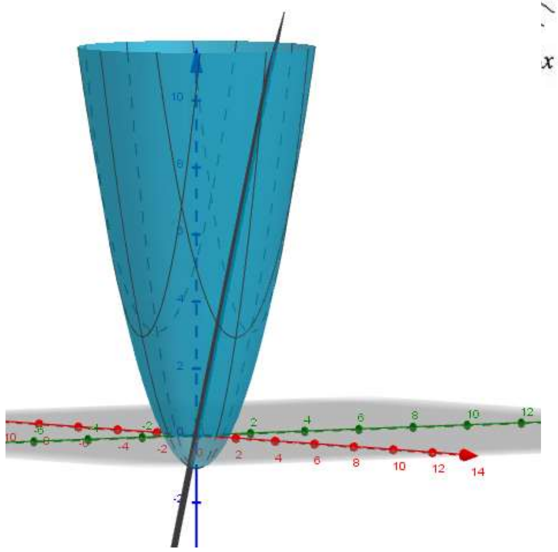


$$3 = \frac{5}{2}\left(\frac{6}{5}\right)$$

$$\lambda = \frac{5}{2}$$

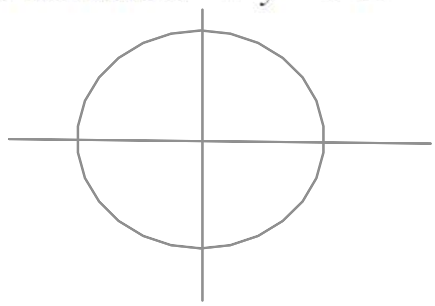
lambda is the ratio between the slopes.





EXAMPLE 3 Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leqslant 1$.

$\nabla f(x, y) = \langle 2x, 4y \rangle$, $\nabla g(x, y) = \langle 2x, 2y \rangle$ $\qquad$ $g(x,y) = x^2 + y^2 - 1$



$2x = \lambda 2x \Rightarrow \lambda = 1$, $2x = 2x$, $x = 0$ If x=0, then $x^2 + y^2 = 1 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

$4y = \lambda 2y$ $\qquad$ $4y = 1(2y) \Rightarrow y = 0$

$\frac{2x}{4y} = \frac{x}{y}$ $\qquad$ $y = 0 : x^2 + 0^2 = 1 \Rightarrow x = \pm 1$

Extrema on the boundary:
$f(0, 1) = 2$, $f(0, -1) = 2$, $f(1, 0) = 1$, $f(-1, 0) = 1$

$f_x = 2x$, $f_y = 2y$
x=0,y=0

$f(0, 0) = 0^2 + 2 \cdot 0^2 = 0$

The max value of f on the region $x^2 + y^2 \leq 1$ is 2 at(0,1),(0,-1) and the minimum value is 0 at (0,0).