Let $f(x,y) = x^2 + y^2 - 4x - 6y + 2.$

Then $f_x(x,y) = 2x - 4 \Rightarrow 2x - 4 = 0 \Rightarrow x = 2$

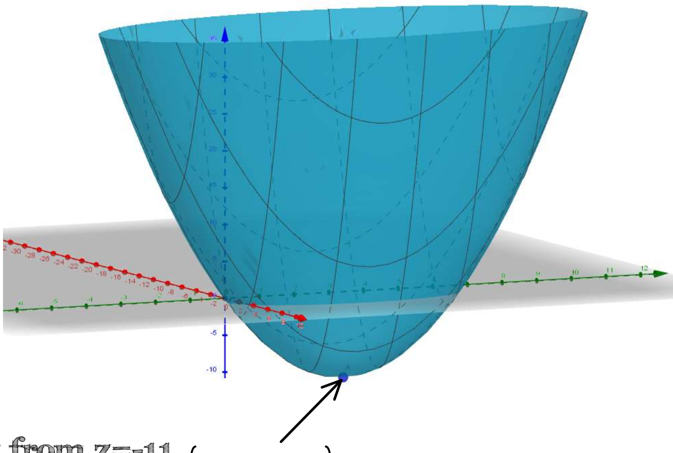
$f_y(x,y) = 2y - 6 \Rightarrow 2y - 6 = 0 \Rightarrow y = 3$

So there is a critical point at (2,3).
If we complete the square, we get

$$f(x,y) = \left(x - \frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2 + \left(y - \frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 2$$

$$f(x,y) = (x-2)^2 - 4 + (y-3)^2 - 9 + 2 = (x-2)^2 + (y-3)^2 - 11$$

Notice that $(x-2)^2 \geq 0$ and $(y-3)^2 \geq 0$, so these make the the values of f just get bigger and bigger, starting from z=-11.

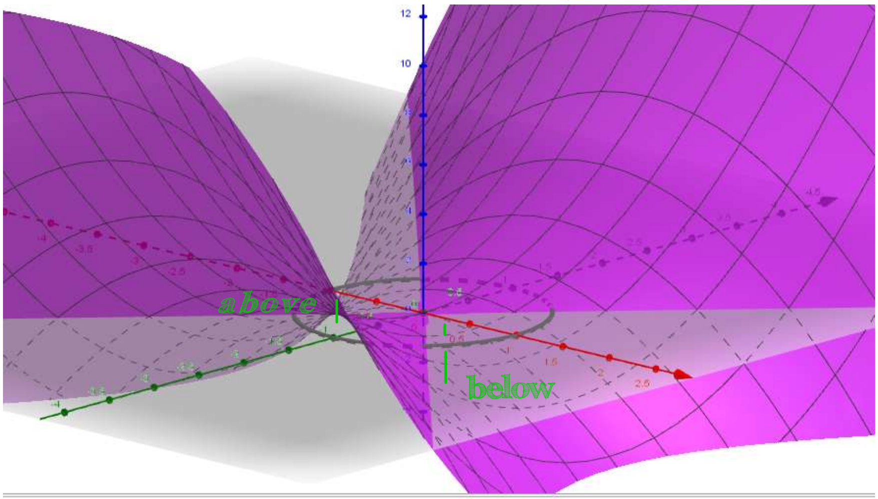


For $f(x,y) = y^2 - x^2$, we get:

$f_x = -2x \Rightarrow -2x = 0 \Rightarrow x = 0$

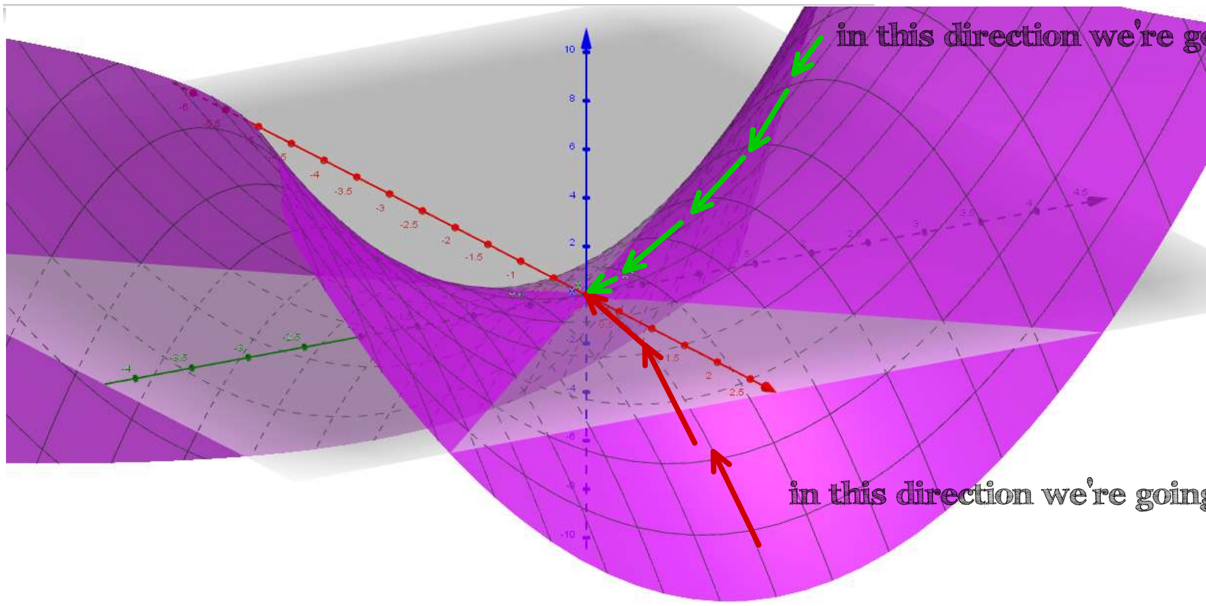
$f_y = 2y \Rightarrow 2y = 0 \Rightarrow y = 0$

When we move along the x axis, we get $f(x,0) = -x^2 < 0, (x \neq 0)$
When we move along the y axis, we get $f(0,y) = y^2 > 0, (y \neq 0)$
So every disk with cener (0,0) contains points giving values of f above nad below the xy plane.
Since this is the case, (0,0) doesn't give an extreme value.



The graph shows one sample disk where there are points such that f>0 and f<0.

Since this surface looks like a saddle, the point (0,0) is called a saddle point of f.



Second Partial Derivatives Test:

Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that f_x (a,b)=0 and f_y (a,b)=0[That is, (a,b) is a critical point of f]. It can be proved using Taylor's Formula that we can study the behavior of f using the following: (Sectoin 14.1 , Thomas Calculus,11th edition has a proof.)

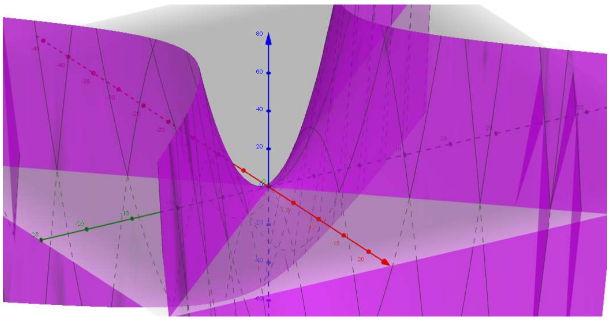
$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$ and the following applies:

i. If D>0 and $f_{xx}(a,b) > 0$, then f(a,b) is a local minimum. (Graph goes up in all directions.)

ii. If D>0 and f_{xx} (a,b)<0, then f(a,b) is a local maximum. (Graph goes down in all directions.)

iii. If D<0 , then f(a,b) is not a local minimum or maximum.

In case iii., (a,b) is called a saddle point of f and the graph of f crosses its tangent plane at (a,b).



If D=0, the test is inconclusive. We could have a saddle point, a minimum or a maximum.

To remember this formula, write in determinant form:

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{yx})^2, \text{ since } f_{yx} = f_{xy}$$

Find the local max. and min. values and saddle points of

$f(x,y) = x^4 + y^4 - 4xy + 1$

$f_x = 4x^3 - 4y \Rightarrow 4x^3 - 4y = 0 \Rightarrow$ divide 4 out $\Rightarrow x^3 = y$ (solve for y)

$f_y = 4y^3 - 4x \Rightarrow 4y^3 - 4x = 0 \Rightarrow$ divide 4 out $\Rightarrow y^3 = x$ (solve for x)

In y^3 =x, replace y with x^3 from the first one, to get $(x^3)^3 = x \Rightarrow x^9 = x$

now solve this:

$x^9 - x = 0$

$x(x^8 - 1) = 0$

$x((x^4)^2 - 1) = 0$

$x(x^4 - 1)(x^4 + 1) = 0$

$x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$

This is zero when x=1, or -1,or 0, from x=0, x^2 -1=0. (x^2 +1) and (x^4 +1) are never 0.

So we have three real roots: -1,0,1 .

To create D(x,y), we need the second partials:

$$f_{xx} = \frac{\partial}{\partial x} f_x = 12x^2$$

$$f_{yy} = \frac{\partial}{\partial y} f_y = 12y^2$$

$$f_{xy} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}(4x^3 - 4y) = -4$$

We can now write

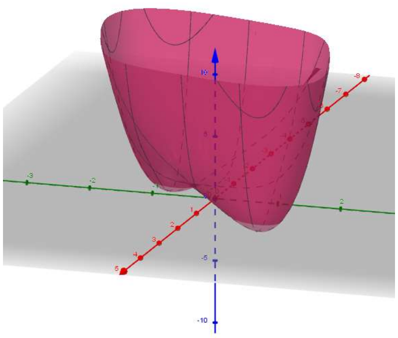
$D(x,y) = (12x^2)\cdot 12y^2 - (-4)^2 = 144x^2y^2 - 16$

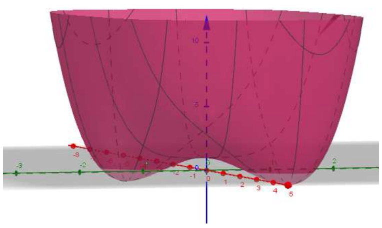
We now test:

D(0,0)= $-16 < 0$, we have that the origin is a saddle point. (no max or min at (0,0))

When x=1, we have y=$(1)^3$ =1, so we have $D(1,1) = 144 - 16 = 128 > 0$, and $f_{xx}(1,1) = 12 > 0$, so we have a local local min at (1,1),where $f(1,1) = 2 - 4 + 1 = -2 + 1 = -1$

When x=-1, we have y=$(-1)^3$ =-1, so we get $D(-1,-1) = 144(-1)^2 - 16 = 144 - 16 = 128 > 0$ and $f_{xx}(-1,-1) = 12(-1)^2 = 12 > 0$, so we get a local minimum again.





As the gaph shows, we have two points (1,1) and (-1,-1) where there are minimums and a saddle point on (0,0).

For both (1,1) and f(-1,-1) we get $f(1,1) = f(-1,-1) = -1$ as the lowest value of z.

Find and classify the critical points of the function

$f(x,y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$

$f_x = 20xy - 10x - 4x^3 \Rightarrow$ factor 2x $\Rightarrow 2x(10y - 5 - 2x^2) = 0$

$f_y = 10x^2 - 8y - 8y^3 \Rightarrow 10x^2 - 8y - 8y^3 = 0 \Rightarrow$ divide 2 away $\Rightarrow 5x^2 - 4y - 4y^3 = 0$

We have to solve this system.

for $2x(10y - 5 - 2x^2) = 0$:

$x = 0$ or $10y - 5 - 2x^2 = 0$

When x=0, $5x^2 - 4y - 4y^3 = 0$

becomes $-4y - 4y^3 = 0$

$-4y(1 + y^2) = 0$

so y=0 from -4y=0

and 1+y^2 is never 0.

Thus, we get the critical point (0,0).

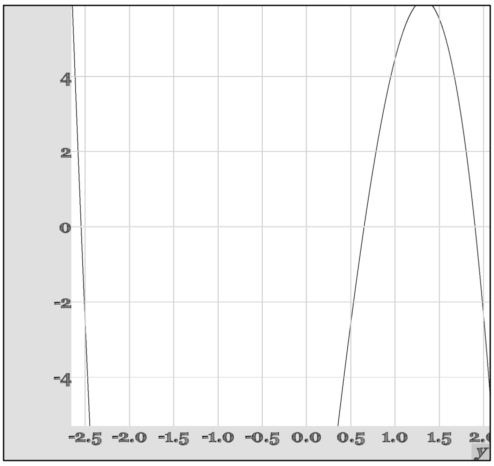
When $10y - 5 - 2x^2 = 0$ we get

$-2x^2 = -10y + 5$

$x^2 = +5y - 2.5$

We can plug this in for x^2 in $5x^2 - 4y - 4y^3 = 0$

to get $5(5y - 2.5) - 4y - 4y^3 = 0$



We can graph this equation and look for the roots to make a reasonable estimate.

We can estimate the roots to be about

y ≈ -2.5452

y=0.6468

and y ≈ 1.8984

Using the values we have on the right from the graph, we get

$x^2 = 5y - 2.5$

$x = \pm\sqrt{5y - 2.5}$

$y \approx -2.5452 \Rightarrow x = \pm\sqrt{5(-2.5452) - 2.5}$

not a real result here

$y \approx 0.6468 \Rightarrow x = \pm\sqrt{5 \cdot 0.6468 - 2.5}$

$= \pm\sqrt{0.7340}$

$= \pm 0.856738$

$y = 1.8984 \Rightarrow x = \pm\sqrt{5 \cdot 1.8984 - 2.5}$

$= \pm 2.644239$

So all our points are the following:

$(0, 0)$

$(\pm 2.644239, 1.8984)$

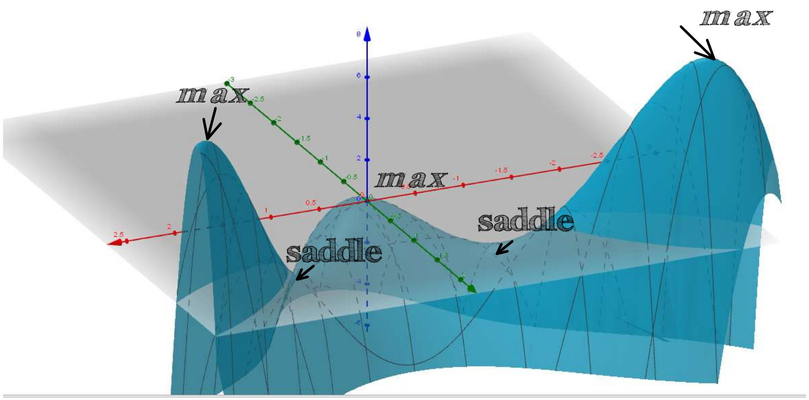
$(\pm 0.856738, 0.6468)$

D(x,y)= $f_{xx} f_{yy} - f_{xy}^2 = (20y - 10 - 12x^2)(-8 - 24y^2) - (20x)^2$

$D(0,0) = (-10)(-8) = 80 > 0$, $f_{xx}(0,0) = -10 < 0$, so we have a local max at (0,0).

$D(\pm 2.644239, 1.8984) = 2488.72 > 0$, $f_{xx}(\pm 2.644239, 1.8984) = -55.93 < 0$, so we have a local max

$D(\pm 0.856738, 0.6468) = -187.64$, so we have a saddle point.



The graph of the surface confirms the math above.

Find the shortest distance from the point (1,0,-2) to the plane x+2y+z=4.

distance from any point (x,y,z) to the point $(1,0,-2)$ is $d=\sqrt{(x-1)^2+(y-0)^2+(z-(-2))^2}$

$\sqrt{(x-1)^2+y^2+(z+2)^2}$

$$=\sqrt{(x-1)^2+y^2+(z+2)^2}$$

Solve for z in the plane:

$z=4-x-2y$

Plug into distance function: d(x,y)= $\sqrt{(x-1)^2+y^2+(4-x-2y+2)^2}=\sqrt{(x-1)^2+y^2+(6-x-2y)^2}$

We can minimize d by minimizing the expression under the root symbol:

$d^2=f(x,y)=(x-1)^2+y^2+(6-x-2y)^2$

$f_x=2(x-1)+2(6-x-2y)(-1)=2x-2-12+2x+4y=4x+4y-14 \quad \Rightarrow f_{xx}=4$

$f_y=2y+2(6-x-2y)(-2)=2y+(12-2x-4y)(-2)=2y-24+4x+8y=4x+10y-24 \quad f_{yy}=10$

$4x+4y-14=0$

$4x+10y-24=0 \Rightarrow -1(4x+10y-24=0) \Rightarrow -4x-10y+24=0 \Rightarrow -4x-10y+24=0$

$4x+4y-14=0$ add down

$-6y=14-24$

$-6y=-10$

$y=\frac{10}{6}=\frac{5}{3}$

Using $x=\frac{5}{3}$, we get $4(x)+4\cdot\frac{5}{3}-14=0$

$4x=-\frac{20}{3}+14$

$4x=\frac{-20}{3}+\frac{42}{3}$

$4x=\frac{22}{3}$

$x=\frac{22}{12}=\frac{11}{6}$

$f_{xy}=\frac{\partial}{\partial y}f_x=\frac{\partial}{\partial y}(4x+4y-14)=4$

Create D(11/6,5/3)= $4\cdot 10-4^2=40-16=24>0$, $f_{xx}(11/6,5/3)=4>0$, so the point $(11/6,5/3)$ gives a local minimum. In this case, the minimum distance is found as follows:

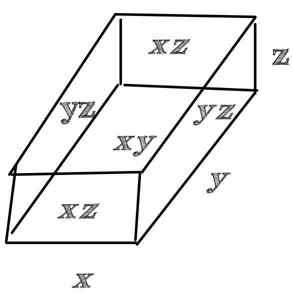
$d(11/6,5/3)=\sqrt{(x-1)^2+y^2+(6-x-2y)^2}=\sqrt{(11/6-1)^2+(5/3)^2+(6-11/6-2(5/3))^2}$ = 2.0412 units.

A rectangular box without a lid is to be made from 12m^2 of cardboard. Find the maximum volume of such a box.

The volume of the box is V=xyz. Just multiply the edge lengths together.

The surface area is $2xz+2yz+xy=12$.

The surface is side+side+bottom+front+back.



We can make the volume into a functio of x, y alone by solving for z in plane:

$2xz+2yz=12-xy$

$z(2x+2y)=12-xy$

$z=\frac{12-xy}{2x+2y}$

So $V=xy\cdot\frac{12-xy}{2x+2y}=\frac{12xy-x^2y^2}{2x+2y}$

So we get $V(x,y)=\frac{2xy-x^2y^2}{2x+2y}$

$f_x=\frac{\partial}{\partial x}\frac{12xy-x^2y^2}{2x+2y}=\frac{\partial}{\partial x}(12xy-x^2y^2)(2x+2y)^{-1}$

$(uv)'=uv'+u'v \quad =(12y-2xy^2)(2x+2y)^{-1}+(12xy-x^2y^2)(-1)(2x+2y)^{-2}(2)$

$=(2x+2y)^{-2}[(12y-2xy^2)(2x+2y)-2(12xy-x^2y^2)]$

$=\frac{(12y-2xy^2)(2x+2y)-2(12xy-x^2y^2)}{(2x+2y)^2}$

$=\frac{24xy+24y^2-4x^2y^2-4xy^3-24xy+2x^2y^2}{(2x+2y)^2}$

$=\frac{-2x^2y^2+24y^2-4xy^3}{(2x+2y)^2}$

$$= \frac{2y^2(-x^2+12-2xy)}{2^2(x+y)^2}$$

$$= \frac{y^2[12-2xy-x^2]}{2(x+y)^2}$$

A lot of similar work shows that $\frac{\partial}{\partial y}\frac{12xy-x^2y^2}{2x+2y} = \frac{x^2(12-2xy-y^2)}{2(x+y)^2}$

Setting these equal to 0 gives us

$\frac{x^2(12-2xy-y^2)}{2(x+y)^2} = 0 \Rightarrow x^2(12-2xy-y^2) = 0 \Rightarrow x = 0$, or $12-2xy-y^2 = 0$

$\frac{y^2[12-2xy-x^2]}{2(x+y)^2} = 0 \Rightarrow y^2(12-2xy-x^2) = 0 \Rightarrow y = 0$, or $12-2xy-x^2 = 0$

With x=0,y=0, we don't have a volume, so
solving for x^2 and y^2 we get

$12-2xy = y^2$

$12-2xy = x^2$

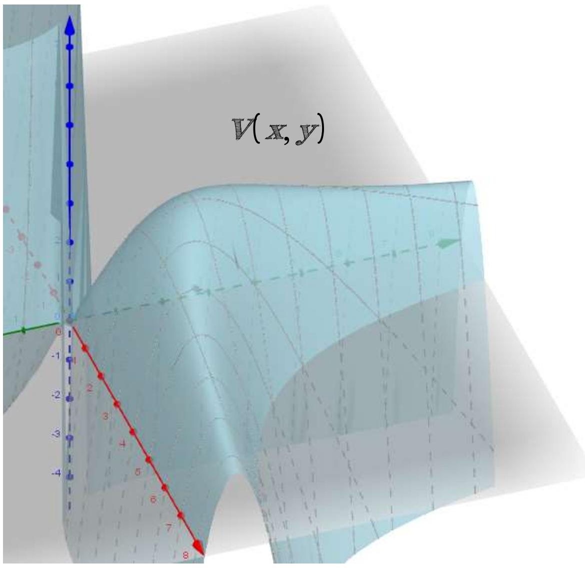
Equating these we get $x^2 = y^2$, and since $x > 0, y > 0$ for physical reasons(it's a box), we get $x = y$. Replacing y with x in $12-2xy-x^2$ =0, we get $12-2x^2-x^2 = 0 \Rightarrow 12-3x^2 = 0 \Rightarrow 3x^2 = 12$

$\Rightarrow x^2 = 4$

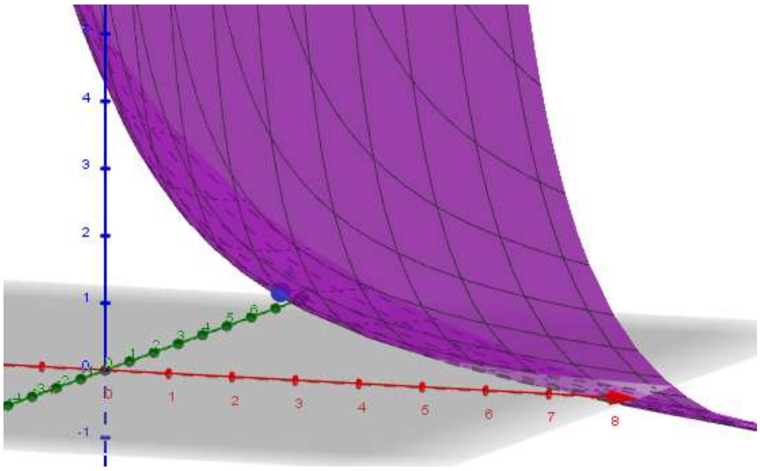
$\Rightarrow x = 2 (-2$ not physical$)$

When $x = 2$, y=2 also. V= $\frac{12xy-x^2y^2}{2x+2y} = \frac{12(2)(2)-2^2\cdot 2^2}{2\cdot 2+2\cdot 2} = \frac{48-16}{8} = \frac{32}{8} = 4$

So we get x=2,y=2,z=1. As the graph of the volume shows, we're at z=1 at x=2,y=2.



$z = \frac{12-xy}{2x+2y}$,z=1 at x=2,y=2



Find the absolute maximum and minimum values of the function
$f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \le x \le 3, 0 \le y \le 2\}$
To answer, we use the extreme value theorem appropriate to multivariable calculus.
Find the values of f at the critical point of D.
Find the extreme values of f on the boundary of D.
The largest of these values from the steps above is the absolute maximum value. The smallest of these values is the absolute minimum.

Find the critical points:

$$f_x = \frac{\partial}{\partial x}(x^2 - 2xy + 2y) = 2x - 2y \Rightarrow 2x = 2y \Rightarrow x = y$$

$$f_y = \frac{\partial}{\partial y}(x^2 - 2xy + 2y) = -2x + 2 \rightarrow -2x + 2 = 0 \Rightarrow -2x = -2 \Rightarrow x = 1$$ Since x=y, y=1 also.

Next look at the boundaries:

Along x=0, $f(0, y) = 0^2 - 2\cdot 0 + 2y = 2y$. This is linear , so it's got a maximum of $f(0, 2) = 2\cdot 2 = 4$.

Along y=0, $f(x, 0) = x^2 - 2x(0) + 2\cdot 0 = x^2$. This is a simple parabola with a maximum at x=3, $f(3, 0) = 9$

Along the boundary where x=3, we get $f(3, y) = 3^2 - 2\cdot 3y + 2y = 9 - 6y + 2y = 9 - 4y$. This is a decreasing function. At $x = 3, y = 2,$ *we* get $f(3, 2) = 9 - 4(2) = 9 - 8 = 1$.

Along the boundary where y=2, we get $f(x, 2) = x^2 - 2x\cdot 2 + 2\cdot 2 = x^2 - 4x + 4$ This is a parabola.

$f'(x, 2) = 2x - 4 \Rightarrow f'(x, 2) = 2x - 4 = 0 \Rightarrow x = 2$ At x=2 ,we get $f(2, 2) = 2^2 - 2\cdot 2\cdot 2 + 2\cdot 2 = 4 - 8 + 4 = 0$

Comparing all these values shows that the minimum is 0 at (2,2) and the maximum is 9 at (3,0).