Let $f(x,y)=x^2+y^2-4x-6y+2$.

Then $f_x(x,y)=2x-4 \Rightarrow 2x-4=0 \Rightarrow x=2$

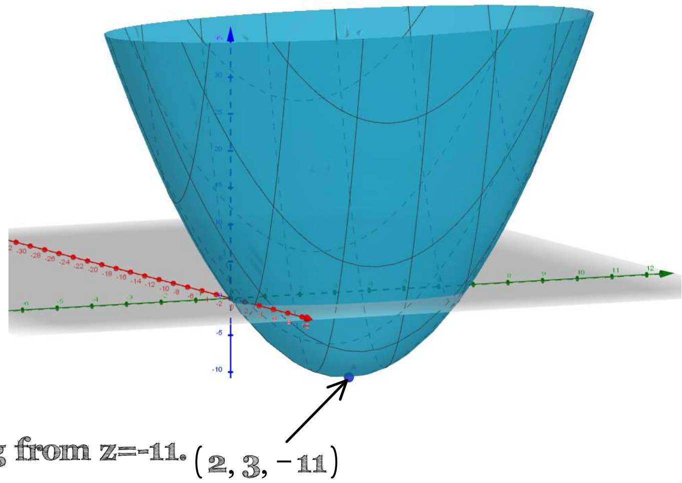
$f_y(x,y)=2y-6 \Rightarrow 2y-6=0 \Rightarrow y=3$

So there is a critical point at (2,3).
If we complete the square, we get

$$f(x,y)=\left(x-\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2+\left(y-\frac{6}{2}\right)^2-\left(\frac{6}{2}\right)^2+2$$

$$f(x,y)=(x-2)^2-4+(y-3)^2-9+2=(x-2)^2+(y-3)^2-11$$

Notice that $(x-2)^2 \geq 0$ and $(y-3)^2 \geq 0$, so these make the the values of f just get bigger and bigger, starting from z=-11.

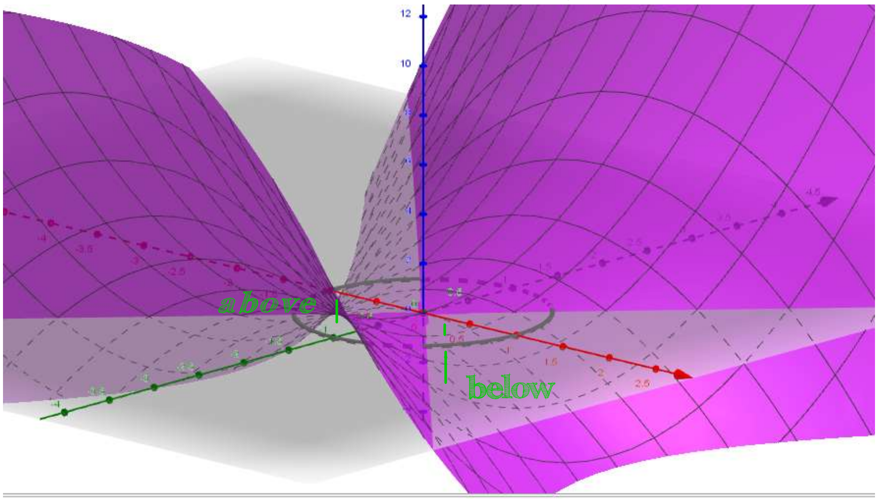


For $f(x,y)=y^2-x^2$, we get:

$f_x=-2x \Rightarrow -2x=0 \Rightarrow x=0$

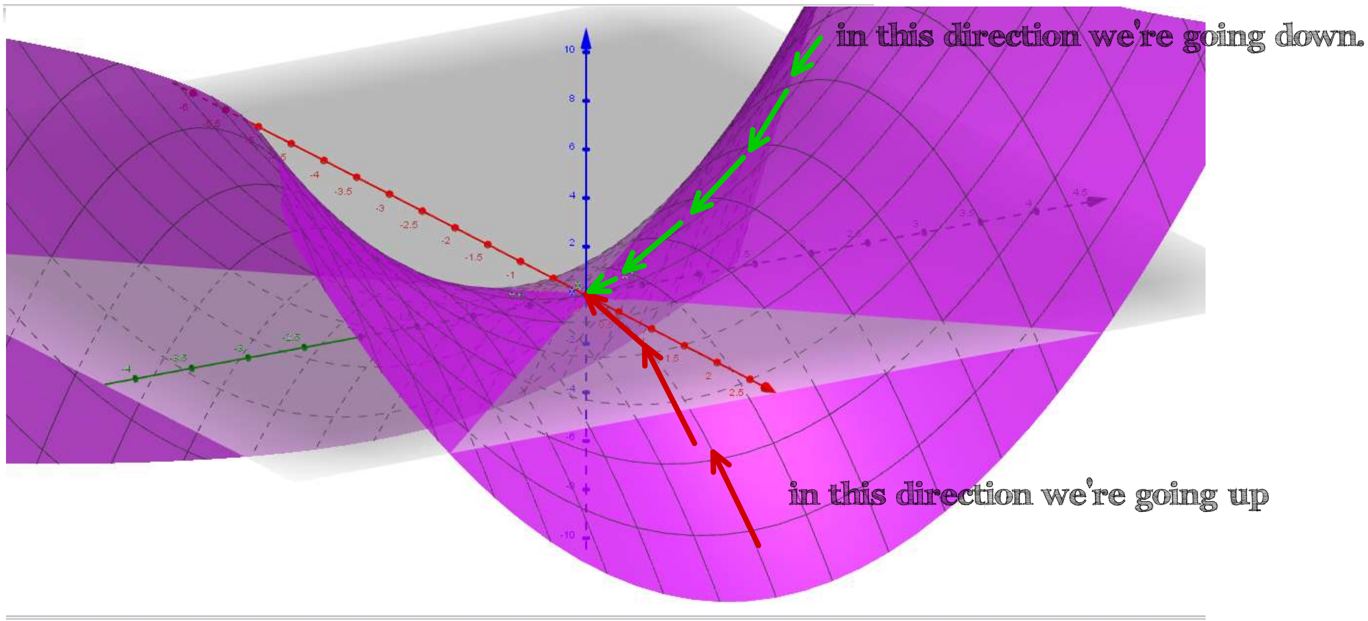
$f_y=2y \Rightarrow 2y=0 \Rightarrow y=0$

When we move along the x axis, we get $f(x,0)=-x^2<0, (x\neq 0)$
When we move along the y axis, we get $f(0,y)=y^2>0, (y\neq 0)$
So every disk with cener (0,0) contains points giving values of f above nad below the xy plane.
Since this is the case, (0,0) doesn't give an extreme value.



The graph shows one sample disk where there are points such that f>0 and f<0.

Since this surface looks like a saddle, the point (0,0) is called a saddle point of f.



Second Partial Derivatives Test:

Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that f_x (a,b)=0 and f_y (a,b)=0[That is, (a,b) is a critical point of f]. It can be proved using Taylor's Formula that we can study the behavior of f using the following: (Sectoin 14.1 , Thomas Calculus,11th edition has a proof.)

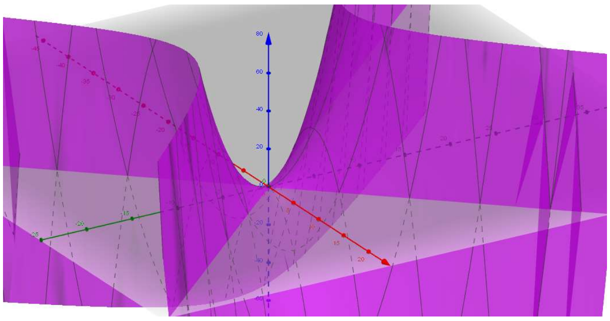
$D(a,b) = f_{xx}(a,b)\,f_{yy}(a,b) - [f_{xy}(a,b)]^2$ and the following applies:

i. If D>0 and $f_{xx}(a,b) > 0$, then f(a,b) is a local minimum. (Graph goes up in all directions.)

ii. If D>0 and f_{xx} (a,b)<0, then f(a,b) is a local maximum. (Graph goes down in all directions.)

iii. If D<0 , then f(a,b) is not a local minimum or maximum.

In case iii., (a,b) is called a saddle point of f and the graph of f crosses its tangent plane at (a,b).



If D=0, the test is inconclusive. We could have a saddle point, a minimum or a maximum.

To remember this formula, write in determinant form:

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{yx})^2, \text{ since } f_{yx} = f_{xy}$$

Find the local max. and min. values and saddle points of

$f(x,y) = x^4 + y^4 - 4xy + 1$

$f_x = 4x^3 - 4y \Rightarrow 4x^3 - 4y = 0 \Rightarrow$ divide 4 out $\Rightarrow x^3 = y$ (solve for y)

$f_y = 4y^3 - 4x \Rightarrow 4y^3 - 4x = 0 \Rightarrow$ divide 4 out $\Rightarrow y^3 = x$ (solve for x)

In y^3 =x, replace y with x^3 from the first one, to get $(x^3)^3 = x \Rightarrow x^9 = x$

now solve this:

$x^9 - x = 0$

$x(x^8 - 1) = 0$

$x((x^4)^2 - 1) = 0$

$x(x^4 - 1)(x^4 + 1) = 0$

$x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$

This is zero when x=1, or -1,or 0, from x=0, x^2 -1=0. (x^2 +1) and (x^4 +1) are never 0.

So we have three real roots: -1,0,1 .

To create D(x,y), we need the second partials:

$$\left.\begin{aligned} f_{xx} &= \frac{\partial}{\partial x} f_x = 12x^2 \\ f_{yy} &= \frac{\partial}{\partial y} f_y = 12y^2 \\ f_{xy} &= \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}(4x^3 - 4y) = -4 \end{aligned}\right\}$$

We can now write

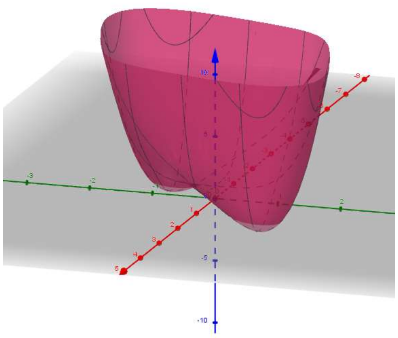
$D(x,y) = (12x^2)\cdot 12y^2 - (-4)^2 = 144x^2y^2 - 16$

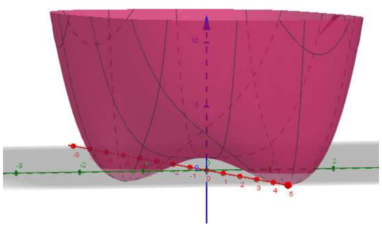
We now test:

D(0,0)= $-16 < 0$, we have that the origin is a saddle point. (no max or min at (0,0))

When x=1, we have y=$(1)^3$ =1, so we have $D(1,1) = 144 - 16 = 128 > 0$, and $f_{xx}(1,1) = 12 > 0$, so we have a local local min at (1,1),where $f(1,1) = 2 - 4 + 1 = -2 + 1 = -1$

When x=-1, we have y=$(-1)^3$ =-1, so we get $D(-1,-1) = 144(-1)^2 - 16 = 144 - 16 = 128 > 0$ and $f_{xx}(-1,-1) = 12(-1)^2 = 12 > 0$, so we get a local minimum again.





As the gaph shows, we have two points (1,1) and (-1,-1) where there are minimums and a saddle point on (0,0).

For both (1,1) and f(-1,-1) we get $f(1,1) = f(-1,-1) = -1$ as the lowest value of z.

Find and classify the critical points of the function

$f(x,y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$

$f_x = 20xy - 10x - 4x^3 \Rightarrow$ factor 2x $\Rightarrow 2x(10y - 5 - 2x^2) = 0$

$f_y = 10x^2 - 8y - 8y^3 \Rightarrow 10x^2 - 8y - 8y^3 = 0 \Rightarrow$ divide 2 away $\Rightarrow 5x^2 - 4y - 4y^3 = 0$

We have to solve this system.

for $2x(10y - 5 - 2x^2) = 0$:

$x = 0$ or $10y - 5 - 2x^2 = 0$

When x=0, $5x^2 - 4y - 4y^3 = 0$

becomes $-4y - 4y^3 = 0$

$-4y(1 + y^2) = 0$

so y=0 from -4y=0

and $1+y^2$ is never 0.

Thus, we get the critical point (0,0).

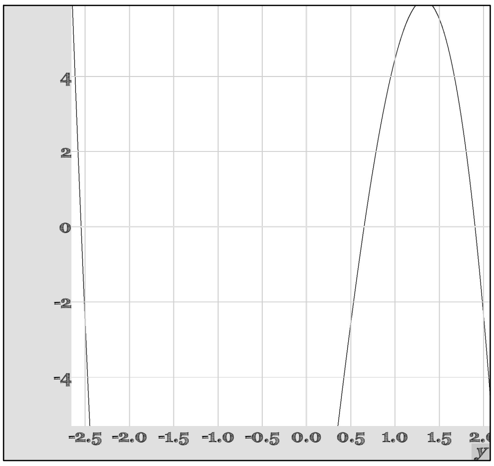
When $10y - 5 - 2x^2 = 0$ we get

$-2x^2 = -10y + 5$

$x^2 = +5y - 2.5$

We can plug this in for x^2 in $5x^2 - 4y - 4y^3 = 0$

to get $5(5y - 2.5) - 4y - 4y^3 = 0$



We can graph this equation and look for the roots to make a reasonable estimate.

We can estimate the roots to be about

y ≈ -2.5452

y=0.6468

and y ≈ 1.8984

Using the values we have on the right from the graph, we get

$x^2 = 5y - 2.5$

$x = \pm\sqrt{5y - 2.5}$

$y \approx -2.5452 \Rightarrow x = \pm\sqrt{5(-2.5452) - 2.5}$

not a real result here

$y \approx 0.6468 \Rightarrow x = \pm\sqrt{5 \cdot 0.6468 - 2.5}$

$= \pm\sqrt{0.7340}$

$= \pm 0.856738$

$y = 1.8984 \Rightarrow x = \pm\sqrt{5 \cdot 1.8984 - 2.5}$

$= \pm 2.644239$

So all our points are the following:

$(0,0)$

$(\pm 2.644239, 1.8984)$

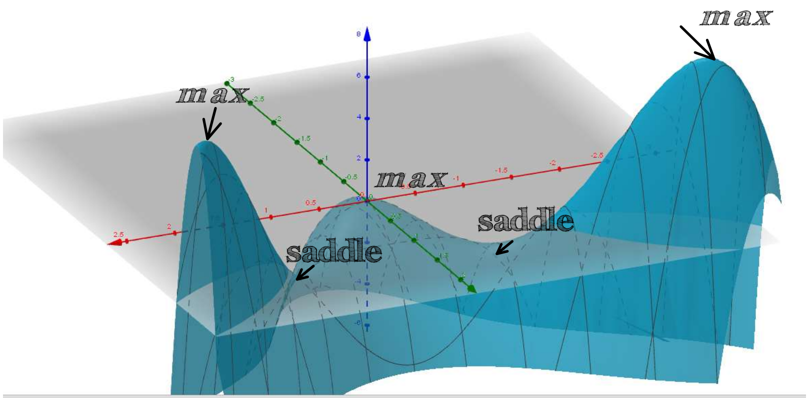
$(\pm 0.856738, 0.6468)$

D(x,y)= $f_{xx}f_{yy} - f_{xy}^2 = (20y - 10 - 12x^2)(-8 - 24y^2) - (20x)^2$

$D(0,0) = (-10)(-8) = 80 > 0$, $f_{xx}(0,0) = -10 < 0$, so we have a local max at (0,0).

$D(\pm 2.644239, 1.8984) = 2488.72 > 0$, $f_{xx}(\pm 2.644239, 1.8984) = -55.93 < 0$, so we have a local max

$D(\pm 0.856738, 0.6468) = -187.64$, so we have a saddle point.



The graph of the surface confirms the math above.

Find the shortest distance from the point (1,0,-2) to the plane x+2y+z=4.

distance from any point (x,y,z) to the point $(1,0,-2)$ is $d=\sqrt{(x-1)^2+(y-0)^2+(z-(-2))^2}$

$\sqrt{(x-1)^2+y^2+(z+2)^2}$

$$=\sqrt{(x-1)^2+y^2+(z+2)^2}$$

Solve for z in the plane:

$z=4-x-2y$

Plug into distance function: d(x,y)= $\sqrt{(x-1)^2+y^2+(4-x-2y+2)^2}=\sqrt{(x-1)^2+y^2+(6-x-2y)^2}$

We can minimize d by minimizing the expression under the root symbol:

$d^2=f(x,y)=(x-1)^2+y^2+(6-x-2y)^2$

$f_x=2(x-1)+2(6-x-2y)(-1)=2x-2-12+2x+4y=4x+4y-14 \quad \Rightarrow f_{xx}=4$

$f_y=2y+2(6-x-2y)(-2)=2y+(12-2x-4y)(-2)=2y-24+4x+8y=4x+10y-24 \quad f_{yy}=10$

$4x+4y-14=0$

$4x+4y-14=0$ add down

$4x+10y-24=0 \Rightarrow -1(4x+10y-24=0) \Rightarrow -4x-10y+24=0 \Rightarrow -4x-10y+24=0$

$-6y=14-24$

$-6y=-10$

$y=\frac{10}{6}=\frac{5}{3}$

Using $x=\frac{5}{3}$, we get $4(x)+4\cdot\frac{5}{3}-14=0$

$$4x=-\frac{20}{3}+14$$

$$4x=\frac{-20}{3}+\frac{42}{3}$$

$$4x=\frac{22}{3}$$

$$x=\frac{22}{12}=\frac{11}{6}$$

$$f_{xy}=\frac{\partial}{\partial y}f_x=\frac{\partial}{\partial y}(4x+4y-14)=4$$

Create D(11/6,5/3)= $4\cdot 10-4^2=40-16=24>0$, $f_{xx}(11/6,5/3)=4>0$, so the point $(11/6,5/3)$ gives a local minimum. In this case, the minimum distance is found as follows:

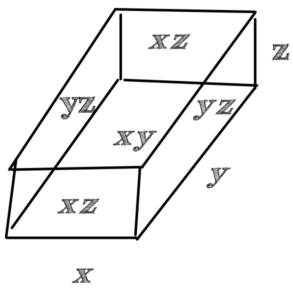
$d(11/6,5/3)=\sqrt{(x-1)^2+y^2+(6-x-2y)^2}=\sqrt{(11/6-1)^2+(5/3)^2+(6-11/6-2(5/3))^2}=2.0412$ units.

A rectangular box without a lid is to be made from 12m^2 of cardboard. Find the maximum volume of such a box.

The volume of the box is V=xyz. Just multiply the edge lengths together.

The surface area is $2xz+2yz+xy=12$.

The surface is side+side+bottom+front+back.



We can make the volume into a functio of x, y alone by solving for z in plane:

$2xz+2yz=12-xy$

$z(2x+2y)=12-xy$

$z=\frac{12-xy}{2x+2y}$

So $V=xy\cdot\frac{12-xy}{2x+2y}=\frac{12xy-x^2y^2}{2x+2y}$

So we get $V(x,y)=\frac{2xy-x^2y^2}{2x+2y}$

$$f_x=\frac{\partial}{\partial x}\frac{12xy-x^2y^2}{2x+2y}=\frac{\partial}{\partial x}(12xy-x^2y^2)(2x+2y)^{-1}$$

$(uv)'=uv'+u'v \quad =(12y-2xy^2)(2x+2y)^{-1}+(12xy-x^2y^2)(-1)(2x+2y)^{-2}(2)$

$$=(2x+2y)^{-2}[(12y-2xy^2)(2x+2y)-2(12xy-x^2y^2)]$$

$$=\frac{(12y-2xy^2)(2x+2y)-2(12xy-x^2y^2)}{(2x+2y)^2}$$

$$=\frac{\cancel{24xy}+24y^2-\cancel{4x^2}y^2-4xy^3-\cancel{24xy}+\cancel{2x^2}y^2}{(2x+2y)^2}$$

$$=\frac{-2x^2y^2+24y^2-4xy^3}{(2x+2y)^2}$$

$$= \frac{2y^2(-x^2+12-2xy)}{2^2(x+y)^2}$$

$$= \frac{y^2[12-2xy-x^2]}{2(x+y)^2}$$

A lot of similar work shows that $\frac{\partial}{\partial y}\frac{12xy-x^2y^2}{2x+2y} = \frac{x^2(12-2xy-y^2)}{2(x+y)^2}$

Setting these equal to 0 gives us

$\frac{x^2(12-2xy-y^2)}{2(x+y)^2} = 0 \Rightarrow x^2(12-2xy-y^2) = 0 \Rightarrow x = 0$, or $12-2xy-y^2 = 0$

$\frac{y^2[12-2xy-x^2]}{2(x+y)^2} = 0 \Rightarrow y^2(12-2xy-x^2) = 0 \Rightarrow y = 0$, or $12-2xy-x^2 = 0$

With x=0,y=0, we don't have a volume, so solving for x^2 and y^2 we get

$12-2xy = y^2$

$12-2xy = x^2$

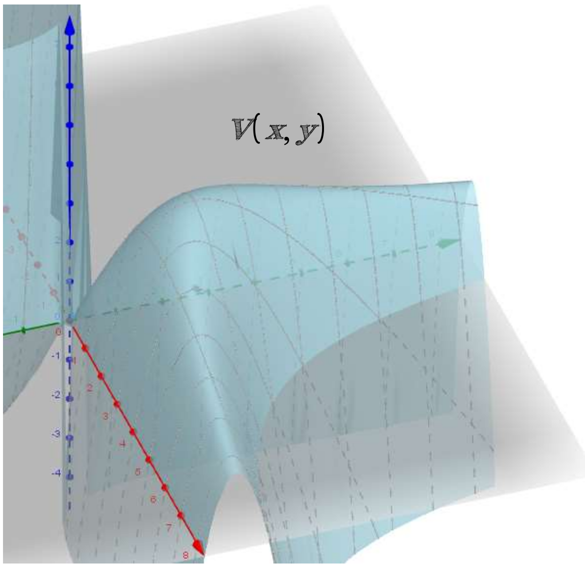
Equating these we get $x^2 = y^2$, and since $x > 0, y > 0$ for physical reasons(it's a box), we get $x = y$. Replacing y with x in $12-2xy-x^2$ =0, we get $12-2x^2-x^2 = 0 \Rightarrow 12-3x^2 = 0 \Rightarrow 3x^2 = 12$

$\Rightarrow x^2 = 4$

$\Rightarrow x = 2\ (-2$ not physical$)$

When $x = 2$, y=2 also. V= $\frac{12xy-x^2y^2}{2x+2y} = \frac{12(2)(2)-2^2\cdot 2^2}{2\cdot 2+2\cdot 2} = \frac{48-16}{8} = \frac{32}{8} = 4$

So we get x=2,y=2,z=1. As the graph of the volume shows, we're at z=1 at x=2,y=2.



$z = \frac{12-xy}{2x+2y}$,z=1 at x=2,y=2