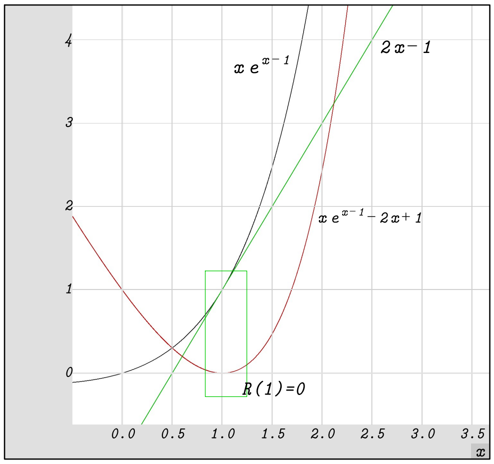
$f(x) = x e^{x-1}$, this is the original function

$f'(x) = (x e^{x-1})' = x' e^{x-1} + x(e^{x-1})' = e^{x-1} + x e^{x-1} = e^{x-1}(1+x)$, differentiate power rule/chain rule

$x_0 = 1.\ L(x) = f(1) + f'(1)(x-1) \Rightarrow L(x) = 1 + 2(x-1) = 1 + 2x - 2 = 2x - 1$, tangent line equation

$R(x) = f(x) - L(x) = x e^{x-1} - (2x-1) = x e^{x-1} - 2x + 1$. Notice that R(1)=0. That is, the error at x=1 is 0.

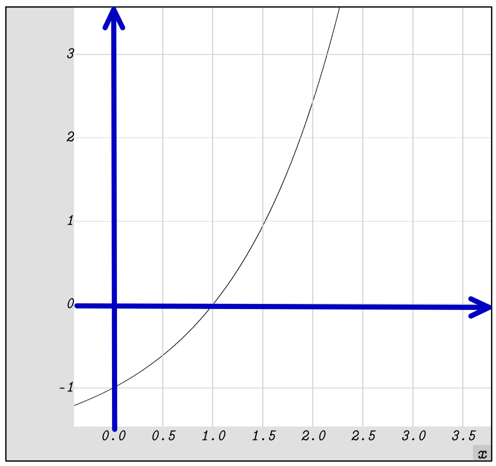


$$\lim_{x \to 1} \frac{R(x)}{x-1}$$

$$\lim_{x \to 1} \frac{x e^{x-1} - 2x + 1}{x-1} \to \text{L'Hopital} \to \lim_{x \to 1} \frac{e^{x-1}(1+x) - 2}{1} = e^{1-1}(1+1) - 2$$

$$= e^0(2) - 2 = 2 - 2 = 0$$

This function , at x=1, is equal to 0.



Why do we use $\lim_{x \to x_0} \frac{R(x)}{x - x_0}$?

$$\lim_{x \to x_0} \frac{f(x) - [f(x_0) + f'(x_0)(x - x_0)]}{x - x_0}$$

b/c $R(x) = f(x) - [f(x_0) + f'(x_0)(x - x_0)]$ = difference in value of function and tangent line approximation

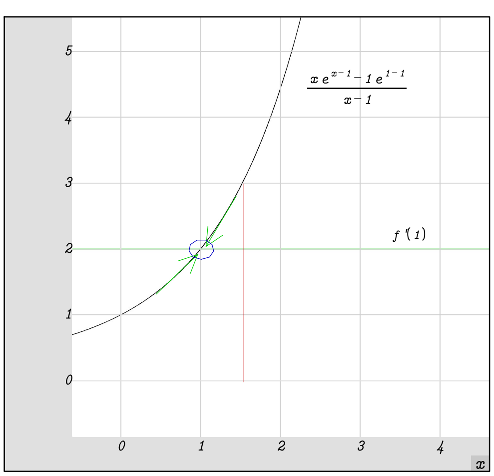
$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0}$$

$$\lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \frac{(x - x_0)}{x - x_0} \right)$$ split apart

$$\underbrace{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}_{\text{average slope}} - \lim_{x \to x_0} f'(x_0)$$ ← notice how $\frac{(x - x_0)}{x - x_0}$ cancels, so we need $x - x_0$ in bottom and not, say, $(x - x_0)^2$

$f'(x_0) - f'(x_0) = 0$ ← This says the average slope, in the limit, is equal to $f'(x_0)$, so the difference is 0.

To relate this to our question, write $\lim_{x \to 1} \frac{x e^{x-1} - [1 e^{1-1} + e^{1-1}(1+1)(x-1)]}{x-1}$, do not simplify



In the graph on the left, the red vertical line shows the diffference in value between $\frac{x e^{x-1} - 1 e^{1-1}}{x-1}$ and $f'(1)$. As x gets closer to 1, the difference between the two graphs goes to 0. In black circle they match.

$$= \lim_{x \to 1} \frac{x e^{x-1} - 1 e^{1-1}}{x-1} - \lim_{x \to 1} \left(e^{1-1}(1+1) \frac{(x-1)}{x-1} \right), \text{ split}$$

$$= \lim_{x \to 1} \frac{x e^{x-1} - 1 e^{1-1}}{x-1} - \lim_{x \to 1} \left(e^{1-1}(1+1) \right) \text{ cancel } \frac{(x-1)}{x-1}$$

$\to$ L'Hopital for limit of $\frac{x e^{x-1} - 1 e^{1-1}}{x-1}$

$$= \lim_{x \to 1} x e^{x-1} - \lim_{x \to 1} f'(1)$$

$$= \lim_{x \to 1} 1 e^{1-1} - \lim_{x \to 1} f'(1) \quad \text{plug in values}$$

$= 1 - 1 = 0$ so it checks!