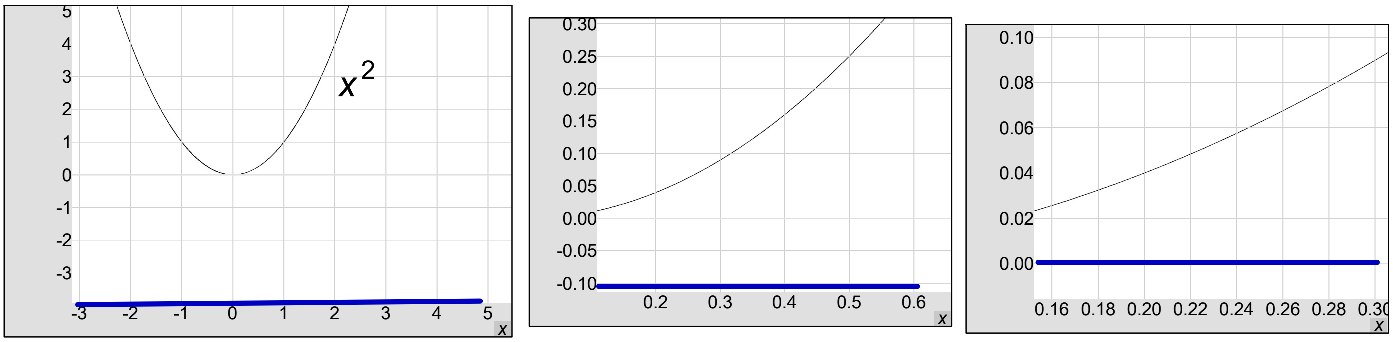
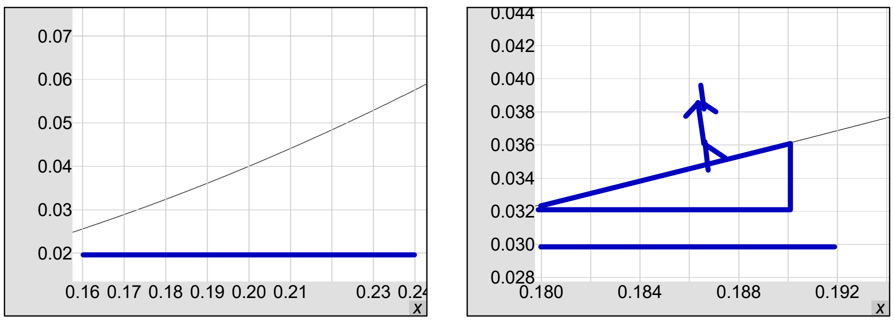
Calculus 1 Notes 9/26, Section 2.7

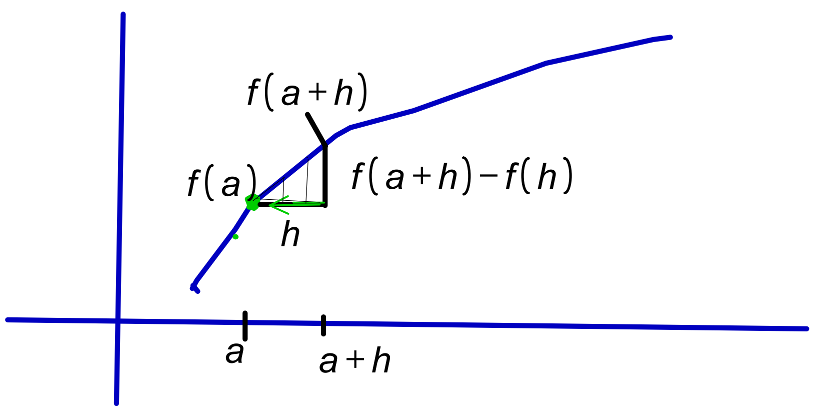
Please be sure your class notes in detail, as I write them, are loaded with your homework solutions.





Once we realize we can zoom in on a curve to make it look like a line, we can find the slope at any point .

$$\frac{0.036-0.032}{0.19-0.180} \approx 0.4$$



Now let h→0!

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \text{slope at x=a!}$$

on the graph of f!

prelim example: $f(x)=x^2$ and find the slope at x=a.

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{(a+h)^2-a^2}{h} = \lim_{h\to 0}\frac{\cancel{a^2}+2ah+h^2-\cancel{a^2}}{h} = \lim_{h\to 0}\frac{2ah+h^2}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{h}(2a+h)}{\cancel{h}}$$

$$= \lim_{h\to 0}(2a+h)$$

$$=2a+0=2a$$

slope formula: $\frac{y_2-y_1}{x_2-x_1}$ (similar logic though)

$2a$ represents the slope on the graph of f at x=a.

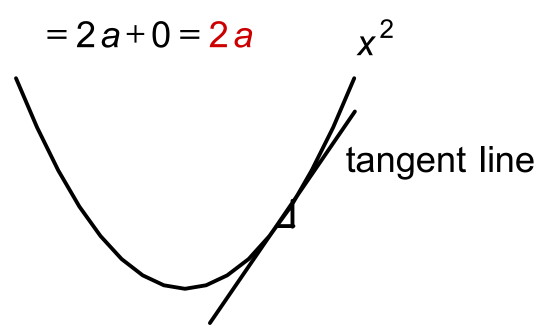
Once we know the slope, we can find the equation of the tangent line. Line that touches the graph at x=a. Let's say we want the equation of the line at x=a=1. slope at x=1 is 2(1)=2. Equation of line $y-y_1=m(x-x_1)$

Our m is 2 . W hen x=1, f(1)= $1^2=1$

$y-1=2(x-1)$ $\qquad$ $y=2x-1 \Leftarrow$ equation of tangent line!

$y-1=2x-2$

$y=2x-2+1$



example 2 in book:

Let $f(x)=\frac{3}{x}$ and find slope at (3,1) and equation of tangent line at that point.

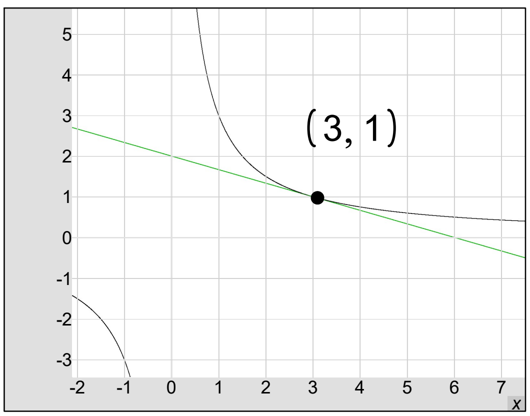
slope at x=3 is $\lim_{h\to 0}\frac{\frac{3}{3+h}-\frac{3}{3}}{h} \xrightarrow{3/3=1} \lim_{h\to 0}\frac{\frac{3}{3+h}-1}{h} \xrightarrow{1\ as (3+h)/(3+h)} \lim_{h\to 0}\frac{\frac{3}{3+h}-\left(\frac{3+h}{3+h}\right)}{h}$

$\xrightarrow{\text{rewrite with common bottom in top}} \lim_{h\to 0}\frac{\frac{3-(3+h)}{3+h}}{h} \xrightarrow{\text{distribute the -1}} \lim_{h\to 0}\frac{\frac{3-3-h}{3+h}}{h}$

$\xrightarrow{\text{cancel off 3 and -3}} \lim_{h\to 0}\frac{\frac{-h}{3+h}}{\frac{h}{1}} \xrightarrow{\text{KEEP, CHANGE,FLIP}} \lim_{h\to 0}\left(\frac{-h}{3+h}\cdot\frac{1}{h}\right) \xrightarrow{cancel\ h} \lim_{h\to 0}\frac{-1}{3+h}$

$\xrightarrow{\text{direct sub.}} \frac{-1}{3+0}=-\frac{1}{3} \Leftarrow$ This represents the slope at x=3 on the graph of 3/x.

Equation of tangent line: $y-1=-\frac{1}{3}(x-3)$



y=mx+b:

$y-1=-\frac{1}{3}x-\frac{1}{3}(-3)$

$y-1=-\frac{1}{3}x+\frac{3}{3}$

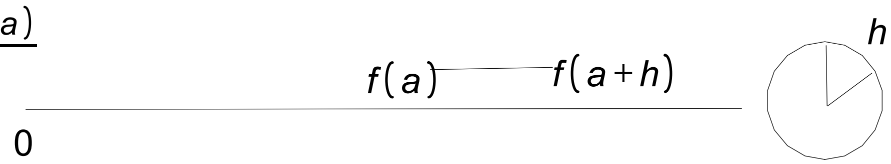
$y-1=-\frac{1}{3}x+1$

$y=-\frac{1}{3}x+2 \Leftarrow$ equation of tangent line

---

velocity: average velocity= $\frac{\text{displacement}}{time}=\frac{f(a+h)-f(a)}{h}$, f represents position of object

velocity at t=a : $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$



instantaneous velocity..velocity at a particular instant in time

displacment=change in position

example 3: Suppose a ball is dropped from the deck of the CN Tower, (tower in Canada) , which is 450meters above ground.

(a) velocity of ball after 5 seconds: position is $f(t)=4.9t^2 \Leftarrow$ (from physics)

We want the velocith at t=5(one instant in time)

$\lim_{h\to 0}\frac{4.9(5+h)^2-4.9\cdot 5^2}{h} \xrightarrow{\text{4.9 factor out}} \lim_{h\to 0}\frac{4.9[(5+h)^2-5^2]}{h} \xrightarrow{\text{pull 4.9 outside}} 4.9\lim_{h\to 0}\frac{(5+h)^2-25}{h}$

$4.9\lim_{h\to 0}\frac{5^2+2(5)h+h^2-25}{h} \xrightarrow{\text{simplify top}} 4.9\lim_{h\to 0}\frac{25+10h+h^2-25}{h}=4.9\lim_{h\to 0}\frac{10h+h^2}{h}$

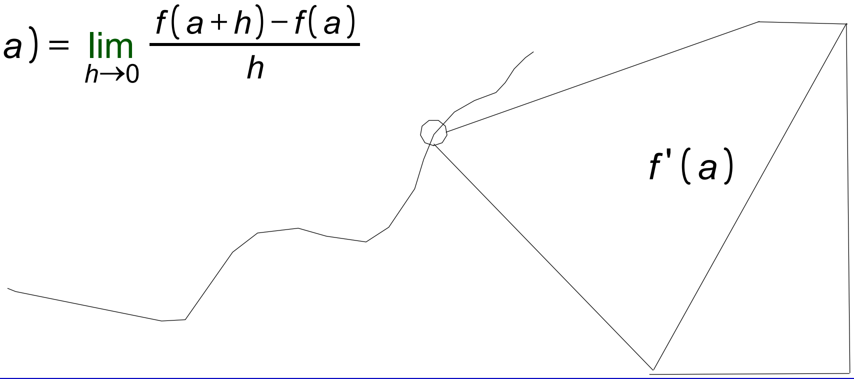
$\xrightarrow{\text{factor h and divide it away}} 4.9\lim_{h\to 0}(10+h) \xrightarrow{\text{direct substitution}} 4.9(10+0)=49m/s$

So at t=5, (only this one instant in time) the object is going at 49 meters per second.

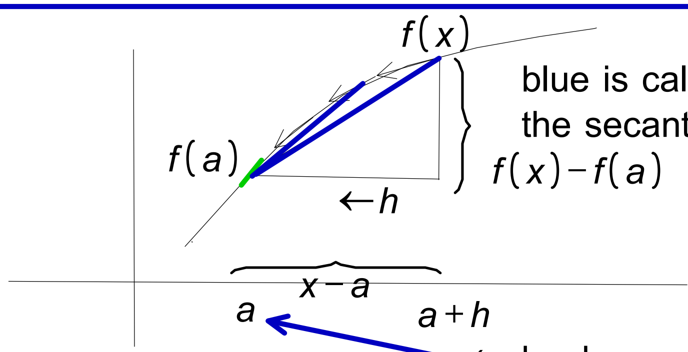
Derivatives:

The derivative of a function f at a number a, denoted by $f'(a)$ ⇐ (f prime of a)

is $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$



zoom in over a very tiny window so the curved function looks like a straight line!



blue is called secant line.. As h gets smaller, eventually the secant line becomes the tangent line.

x ⇐ book now calls this x.. x=a+h $\to x-a=h$

We can now write $\frac{f(x)-(a)}{x-a}$

$\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

Now look at picture...x is going towards a, so we write $x \to a$ (not $h \to 0$)

example 4: Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number x=a.

Doing it by the defintion: $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

Limit is with respect to h ($h \to 0$), so anything without h stays!

$$= \lim_{h \to 0} \frac{(a+h)^2 - 8(a+h) + 9 - 1[a^2 - 8a + 9]}{h}$$

$$= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h}$$ (expand, distribute -1 and -8)

$$= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} \xrightarrow{\text{factor h out}} \lim_{h \to 0} \frac{h(2a+h-8)}{h} \xrightarrow{\text{cancel h}} \lim_{h \to 0} (2a+h-8)$$

$$\xrightarrow{\text{direct sub.}} 2a + 0 - 8 = 2a - 8$$ ⇐ leave a and -8 b/c neither has h!

The slope of x^2 -8x+9 at x=a is 2a-8!! Using this find equation of tangent line at (3,-6)

Evaluate f prime at x=3: $f'(3) = 2 \cdot (3) - 8 = 6 - 8 = -2$ ⇐ goes into tangent line equation!!

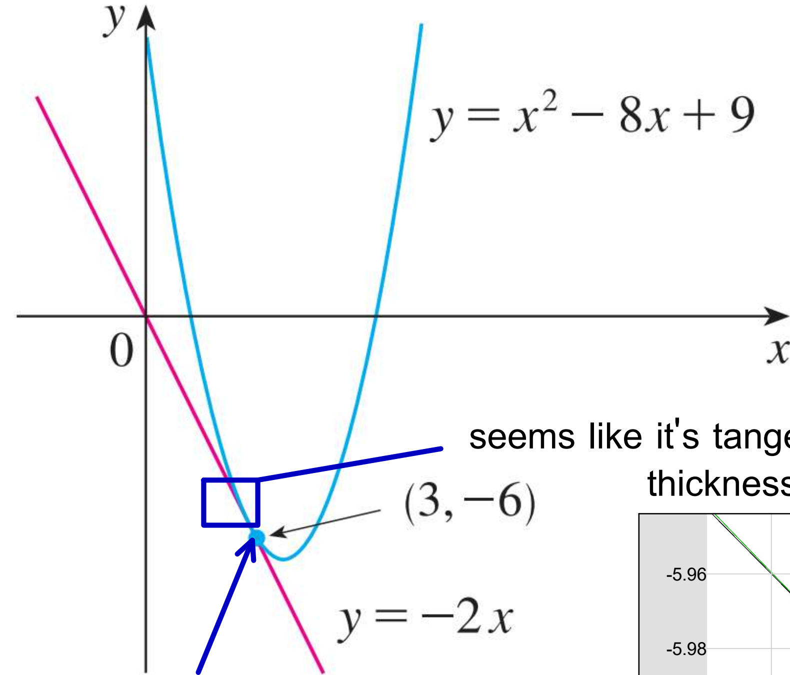
-2 is found but the point (3,-6) is given!

$y - (-6) = -2(x-3)$ ⇐ plug into $y - y_1 = m(x - x_1)$ where m=f'(a)

$y + 6 = -2x + 6$

$y = -2x + 6 - 6$

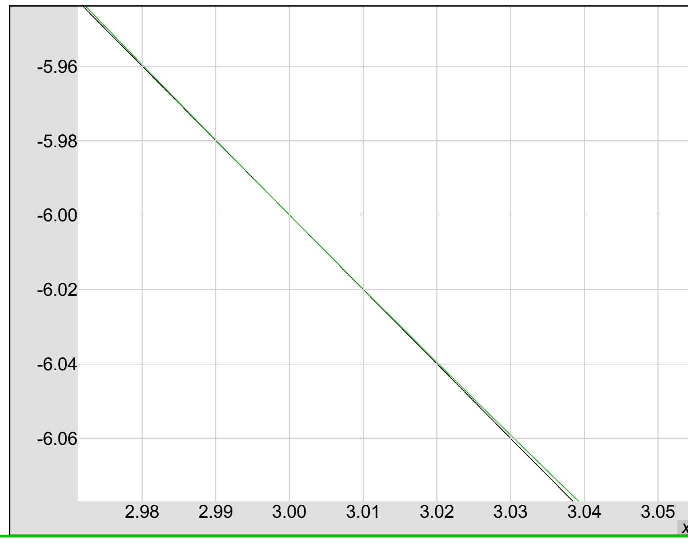
$y = -2x$ ⇐ equation of tangent line to the graph of $x^2 - 8x + 9$



seems like it's tangent here..not true..it's just the thickness!
thickness makes things visible!!∞

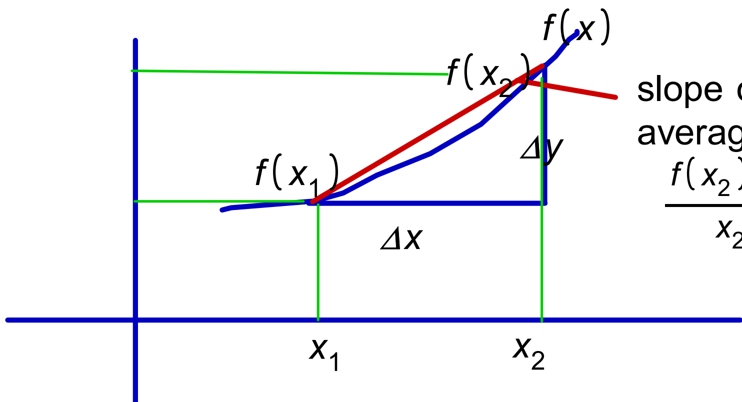
only real point of tangency!

tiny
view
window!!



Rates of Change:

slope= rate of change= $\dfrac{\Delta y}{\Delta x} = \dfrac{\text{change in y coords.}}{\text{change in x coords.}} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(x+h) - f(x)}{h}$ ⇐ no limit!!



slope of red line(secant line) is called the average rate of change of f.

$\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ = average rate of change

Δy is not $\Delta \cdot$ y
it's "delta y" as one, unbreakable symbol!

Δx is like h(single symbol)

Practical Example/Example 6:
We're making bolts of fabric with a fixed width. The cost of producing x yards of this fabric is C=f(x). f(5)=cost to make 5 yards of fabric!!(not given an expression with x)

(a) What is the meaning of f'(x)? What are its units?
answer:The derivative gives us the instantaneous rate of change of the cost. More fully: rate of change of production cost with respect to the number of yards produced
"with respect to " ..before it put the stuff on the vertical..after it put the indepdent variable

In symbols, $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}$ (ΔC=change in cost, Δx=change in yards of fabric)

Here, we're using Δx *to* mean h, so we write

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}$$

units: *cost* is in dollars, x is in yards, so rate of change units= $\frac{\text{dollars}}{yards}$

"dollars per yard"

(b) $f'(1000) = 9 \Leftarrow$ what does this mean?
x=1000, 9=9/1 9$/1yard
Product cost is increasing at a rate of 9$ per yard! Only at x=1000 is the value 9!
At x=1001 it could already be different.

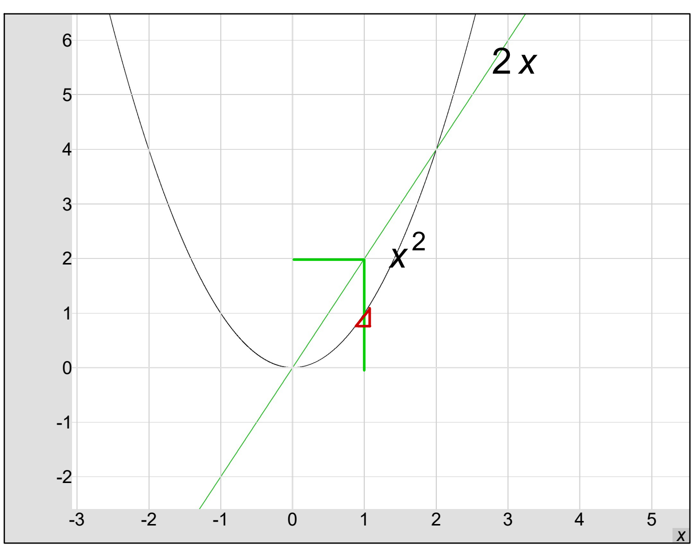
$f(x) = x^2$

$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - h^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0}(2x+h)$

$= 2x + 0 = 2x \Leftarrow$ this is a function of x!!(not constant!!!)

$f'(1) = 2 \cdot 1 = 2 \Leftarrow$ On the graph of x^2 at x=1 the slope is 2.
f'(2)= $2 \cdot 2 = 4 \Leftarrow$ *On* the graph of x^2 at x=2 the slope is 4. (different from 2 above)

$f'(-1) = 2(-1) = -2 \Leftarrow$ On the graph of x^2 at x=-1 the slope is =-2!
$2 \neq 4 = -2 \cdot$ (the slope is a changing quantity!!)



The y-coordinate at x=1 of 2x is the slope on the graph of x^2 at x=1!

Slope if found from red slope triangle at x=1 would be the y-coordinate on green graph at x=1!