Section 4.5/More Curve Sketching:

Make sure in your homework solutions include a sketch.

Page 320: Sketch $f(x)=\frac{x^3}{x^2+1}$

1 domain: $x \in \mathbf{R}$ (x^2+1 is never 0)

2 x intercept(s):

$$f(x)=0 \Rightarrow \frac{x^3}{x^2+1}=0 \Rightarrow x^3=0(x^2+1) \Rightarrow x^3=0 \Rightarrow x=0$$

3. y intercept: $f(\mathbf{0})=\frac{0^3}{0^2+1}=\frac{0}{0+1}=\frac{0}{1}=0 \quad \Rightarrow (0,0)$

4. vertical asymptotes? $x^2+1=0$

$x^2=-1$

no sol. over the real numbers, no VA

5. slant asymptote: $x^2+1\overline{)x^3+0x^2+0x+0}$ with quotient $\mathbf{x}$ (stuff in red parts) $x^3/x^2=x$

$$\begin{array}{l} -x^3 \quad -x \\ \hline \quad\quad\; -x \end{array} \qquad \frac{-x}{x^2}=-\frac{1}{x} \Leftarrow \text{stop!!}$$

$f(x)=\mathbf{x}-\frac{x}{x^2+1}$ As x goes to infinity, the function behaves like $\mathbf{x}$. — book approach or

$$\lim_{x\to\infty}\frac{x^3}{x^2+1}=\lim_{x\to\infty}\frac{x^3}{x^2}=\lim_{x\to\infty}\mathbf{x}=\infty$$

6. $f'(x)=\left(\frac{x^3}{x^2+1}\right)'=\frac{loDhi-hiDlo}{lo^2}=\frac{(x^2+1)(3x^2)-x^3(2x)}{(x^2+1)^2}=\frac{3x^4+3x^2-2x^4}{(x^2+1)^2}=\frac{x^4+3x^2}{(x^2+1)^2}=\frac{x^2(x^2+3)}{(x^2+1)^2}$

$-\infty \ldots \leftarrow$ -------- 0 ---------$\rightarrow \ldots \infty$

$x=-1$ $\quad$ $x=1$

$$\frac{x^2(x^2+3)}{(x^2+1)^2}=0$$

$$x^2(x^2+3)=0(x^2+1)^2$$

$$x^2(x^2+3)=0 \qquad x^2+3>0 \text{ always!!!}$$

$$x^2\left(\frac{x^2+3}{x^2+3}\right)=\frac{0}{x^2+3}$$

$$x^2=0$$

$f(-1)=\frac{(-1)^2((-1)^2+3)}{((-1)^2+1)^2}$

$=\frac{1(1+4)}{(1+1)^2}>0$ f is increasing over $(-\infty,\ 0)$

$f(1)=\frac{(1)^2(1^2+3)}{(1^2+1)^2}>0$ so f is increasing over $(0,\infty)$

7. local max or min? $x=0$ from f'=0. For x<0, f'>0. For x>0, f'>0, so there is no local max or min. no $\cup$ or $\cap$.

8. $f''(x)=\left(\frac{x^2\bullet(x^2+3)}{(x^2+1)^2}\right)'=\left(\frac{(x^4+3x^2)}{(x^2+1)^2}\right)'$ (power rule is easier on x^4+3x^2 than product on $x^2(x^2+3)$)

$$=\frac{(x^2+1)^2(4x^3+6x)-(x^4+3x^2)\,[(x^2+1)^{\mathbf{2}}]'}{((x^2+1)^2)^2}=\frac{(x^2+1)^2(4x^3+6x)-(x^4+3x^2)[2(x^2+1)\cdot 2x]}{((x^2+1)^2)^2}$$

$=\ldots$ *some* algebra steps $=\frac{2x(3-x^2)}{(x^2+1)^3}$ (bottom is never zero b/c $x^2+1>0$) $\qquad \sqrt{3}\approx 1.73$

$\frac{2x(3-x^2)}{(x^2+1)^3}=0$

concavity: $-\infty \ldots \leftarrow$ ----- CU $-\sqrt{3}$ ------- CD 0 ------- CU $\sqrt{3}$ ------- CD $> +\infty$

$2x(3-x^2)=0(x^2+1)^3$

$2x(3-x^2)=0$

$2x=0 \quad 3-x^2=0$

$x=0 \quad 3=x^2$

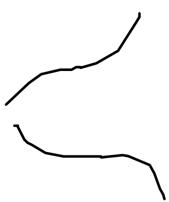
$x=\pm\sqrt{3}$

$x=-2$: $\frac{2(-2)(3-(-2)^2)}{((-2)^2+1)^3}>0$ (CU)

$x=-1$: $\frac{2(-1)(3-(-1)^2)}{((-1)^2+1)^3}<0$

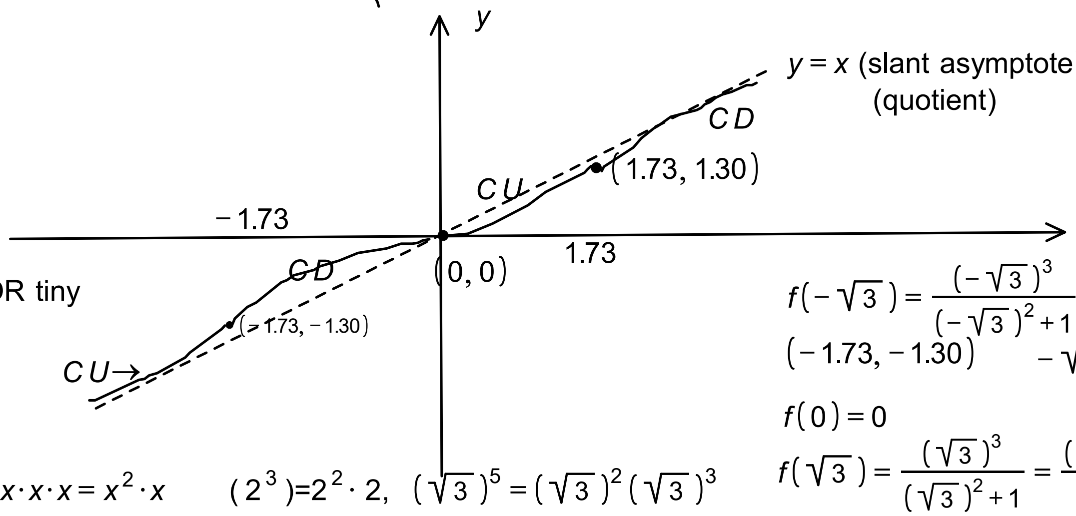
$x=1$: $\frac{2\cdot 1(3-1^2)}{(1^2+1)^3}>0$

$x=2$: $\frac{2\cdot 2(3-2^2)}{(2^2+1)^2}<0$



topology

inflection points: $(-\sqrt{3}, f(-\sqrt{3})), (0, f(0)), (\sqrt{3}, f(\sqrt{3}))$



Must HONOR tiny details:)))

$$f(-\sqrt{3})=\frac{(-\sqrt{3})^3}{(-\sqrt{3})^2+1}\approx -1.30$$

$(-1.73, -1.30) \qquad -\sqrt{3}=-1.73$

$f(0)=0$

$$f(\sqrt{3})=\frac{(\sqrt{3})^3}{(\sqrt{3})^2+1}=\frac{(\sqrt{3})^2\sqrt{3}}{3+1}=\frac{3\sqrt{3}}{4}$$

$x^3=x\cdot x\cdot x=x^2\cdot x \qquad (2^3)=2^2\cdot 2, \quad (\sqrt{3})^5=(\sqrt{3})^2(\sqrt{3})^3$

example 2: $f(x)=\dfrac{x^2+6x+8}{x+5}$ rational function $\dfrac{N(x)}{D(x)}$ where N and D are both polys.

$$f'=\left(\frac{x^2+6x+8}{x+5}\right)'=\frac{lo\,D\,hi-hi\,D\,lo}{lo^2}=\frac{(x+5)(2x+6)-(x^2+6x+8)(1)}{(x+5)^2}=\frac{2x^2+6x+10x+30-x^2-6x-8}{(x+5)^2}=\frac{x^2+10x+22}{(x+5)^2}$$

critical values c: $\dfrac{x^2+10x+22}{(x+5)^2}=0\Rightarrow x^2+10x+22=0,\ a=1,\ b=10,\ c=22$

$$x=\frac{-10\pm\sqrt{(10)^2-4(1)(22)}}{2\cdot 1}=\frac{-10\pm\sqrt{100-88}}{2}=\frac{-10\pm\sqrt{12}}{2}$$
$$=\frac{-10\pm\sqrt{4\cdot 3}}{2}=\frac{-10\pm\sqrt{4}\sqrt{3}}{2}=\frac{-10\pm 2\sqrt{3}}{2}=-5\pm\sqrt{3}$$

$x=-5+\sqrt{3},\ x=-5-\sqrt{3}$
$\approx -3.27 \qquad \approx -6.73$

increasing over...? ????

-5 ↓

$-\infty$ ------------------ -6.73 ------------------ -3.27--------- ∞

$x=-7$: $\dfrac{(-7)^2+10(-7)+22}{(-7+5)^2}$ — $0.25>0$ — f increases on $(-\infty,-6.73)$

$x=-4$: $\dfrac{(-4)^2+10(-4)+22}{(-4+5)^2}=-2<0$ — decreases over $(-6.73,-3.27)$

$x=0$: $\dfrac{0^2+10\cdot 0+22}{(0+5)^2}=\dfrac{22}{25}>0$ — f increases over $(-3.27,\infty)$

b/c -5 plugged into f' makes it undefined!

between -6.73 and -5:
6:
$$x=\frac{(-6)^2+10(-6)+22}{(-6+5)^2}=$$
$=-2<0$

Here x=-5 makes the first derivative undefined, so it must be included as a critical value.

answer in MOM: $(-\infty,-6.73)\cup(-3.27,\infty)$

decreasing part: $(-6.73,-5)\cup(-5,-3.27)$

$f\uparrow$ | $f\downarrow$ | $f\downarrow$ | $f\uparrow$

$-\infty\ldots$ — $-5-\sqrt{3}$ — -5 — $-5+\sqrt{3}$ — ∞

root of f' — $x=-5$ makes f' undefined! — root of f'

second derivative: $f''(x)=(f'(x))'=\left(\dfrac{x^2+10x+22}{(x+5)^2}\right)'=\dfrac{(x+5)^2(2x+10)-(x^2+10x+22)\cdot 2(x+5)(1)}{((x+5)^2)^2}$

$(a+b)^2=a^2+2ab+b^2$

~~$(a+b)^2=a^2+b^2$~~

$$=\frac{(x+5)^2(2x+10)-(x^2+10x+22)(2x+10)}{(x+5)^4}$$
$$=\frac{(2x+10)[(x+5)^2-(x^2+10x+22)]}{(x+5)^4}$$
$$=\frac{(2x+10)[x^2+10x+25-x^2-10x-22]}{(x+5)^4}$$
$$=\frac{(2x+10)[25-22]}{(x+5)^4}=\frac{3(2x+10)}{(x+5)^4}=\frac{3\cdot 2(x+5)}{(x+5)^4}=\frac{6(x+5)}{(x+5)^4}=\frac{6}{(x+5)^3}$$

local max: look at $x=-5-\sqrt{3}$ (-6.73)

There is a local maxy of $f(-6.73)$ at x= -6.73
-7.464

local min of $f(-3.27)$ at x= -3.27
-0.535898

h. f is concave up on $\dfrac{6}{(x+5)^3}\Leftarrow$ undefined at x= -5

CD | CU

$-\infty$ ------------------- -5 --------------- ∞

$x=-6$: $\dfrac{6}{(-6+5)^3}$ — $-6<0$

$x=0$: $\dfrac{6}{(0+5)^3}>0$

concave up over $(-5,\infty)$
concave down over $(-\infty,-5)$

vertical asymptote: look at where f is undefined:

$f(x)=\dfrac{x^2+6x+8}{x+5}$, $x=-5$ is the VA.

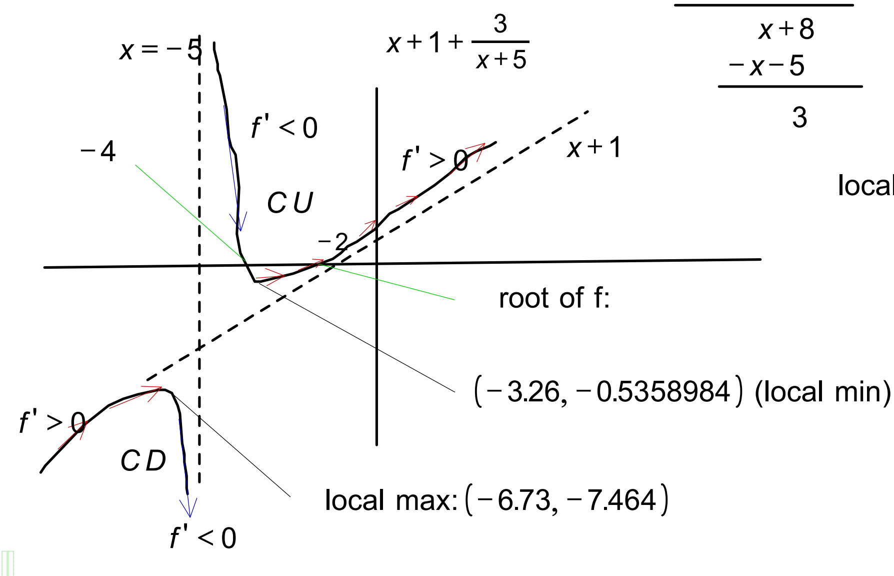
inflection point: $f''(x)$ is undefined x=-5

$x=-5$ is not a ROOT of f''. It just makes it undefined.

slant asymptote: $y=\dfrac{x^2+6x+8}{x+5}$

$$\begin{array}{r} x+1 \\ x+5\,\overline{\smash{)}\,x^2+6x+8} \\ -x^2-5x \end{array}$$

$\dfrac{x^2}{x}=x \qquad \dfrac{x}{x}=1$



$x+8$

$-x-5$

3

slant asymptote equation

quotient: $y = x+1$

local min:

$f(x) = 0$

$$\frac{x^2+6x+8}{x+5} = 0$$

$$x^2+6x+8=0$$

$$(x+2)(x+4)=0$$

$$x=-2, x=-4$$