Math 200 notes 11/15/2023. Please put away phones and computers and take **meticulous** notes.

Section 6.2/Compound Interest:

Annual: once per year, Semiannual: twice per year, Quarterly: 4 times per year, Monthly: 12 times per year, Daily: 365 times per year
If the interest due at the end of each period is added to the principal we call this compounding.

Example 1/Page 302: A bank pays interest of 4% per annum compounded quarterly. If 200 is deposited in an account and the quarterly interest is left in the account, how much money is in the account after 1 year?

$I = Prt$:

end of first quarter (3 months): I= $200 \cdot 0.04 \cdot \left(\frac{3}{12}\right) = \$2.00$. New principal= $200 + 2 = 202$.

end of second quarter (next 3 months:) $I = 202 \cdot 0.04 \cdot \frac{3}{12} = \$2.02$. New principal= $202 + 2.02 = 204.02$

end of third quarter (next 3 months) $I = 204.02 \cdot 0.04 \cdot \frac{3}{12} = \$2.04$. New principal= $204.02 + 2.04 = 206.06$

end of fourth quarter (last 3 monhts) $I = 206.06 \cdot 0.04 \cdot \frac{3}{12} = 2.06$. New pricipal = 208.12

For each part above we have $\frac{3}{12} = \frac{3 \cdot 1}{3 \cdot 4} = \frac{1}{4} \Leftarrow$ 4 is the number of times we compound.

observations:

$I = Pr \cdot \frac{1}{n} = P \cdot \frac{r}{n}$

A= $P + I = P + P \cdot \frac{r}{n} = P\left(1 + \frac{r}{n}\right)^{1}$

A= $P\left(1 + \frac{r}{n}\right) \cdot 1 + P\left(1 + \frac{r}{n}\right)\frac{r}{n}$

$P(1 + r/n)$ is common so factor it out:

$A = P\left(1 + \frac{r}{n}\right)\left[1 + \frac{r}{n}\right] = A\left(1 + \frac{r}{n}\right)^2$

$A = P\left(1 + \frac{r}{n}\right)^2 \cdot 1 + P\left(1 + \frac{r}{n}\right)^2 \frac{r}{n}$

$A = P\left(1 + \frac{r}{n}\right)^2\left[1 + \frac{r}{n}\right] = P\left(1 + \frac{r}{n}\right)^3$

$\Rightarrow$ So after n periods we get $A = P\left(1 + \frac{r}{n}\right)^n$

If we repeat this process for t year, we end up with $A = P\left(1 + \frac{r}{n}\right)^{nt}$

Discrete compounding formula: $P = princpal$, $r = rate$, $n =$ number of times we compound, and t=time.

example 2/page 304: If \$1000 is borrowed at a rate of 10% and no payments are made on this loan, what is the amount after 5 years if the compounding takes place:

annual n=1: $A = 1000\left(1 + \frac{0.1}{1}\right)^{1 \cdot 5} = 1000(1 + 0.1)^5 = \$1610.51$ $\Leftarrow$ interest= $A - P = 1610.51 - 600 = \$610.51$

monthly n=12: $A = 1000\left(1 + \frac{0.1}{12}\right)^{12 \cdot 5} = 1000(1 + 0.0083)^{60} = 1645.31 \Leftarrow$ interest= $A - P = 1645.31 - 1000 = \$645.31$

daily n=365: $A = 1000\left(1 + \frac{0.1}{365}\right)^{365 \cdot 5} = 1000(1.64861)^{1825} = 1648.61 \Leftarrow$ interest= $A - P = 1648.61 - 1000 = \$648.61$

observations based on example above:

Per Annum Rate:	Compounding Method	Interest Rate per Period: (effective interest rate)
10%	Annual	0.1
10%	monthly	$0.1/12 = 0.00833$ $\Leftarrow$ at end of each month rate is not 10%
10%	Daily	$0.1/365 = 0.000274 \Leftarrow$ at the end of each day rate is not 10%

Example 3: Investing 10,000 at an annual rate of 4% compounded annually, semiannually, quarterly, monthly and daily will yield the following amounts after 1 year:

Annual compounding (n=1) $A = P\left(1 + \frac{r}{n}\right)^{nt} = 10000\left(1 + \frac{0.04}{1}\right)^{1 \cdot 1} = 10000(1 + 0.04) = \$10,400.$ $t = 1, n = 1$

semiannual compounding (n=2) A= $P\left(1 + \frac{r}{2}\right)^{2 \cdot 1} = 10000\left(1 + \frac{0.04}{2}\right)^2 = \$10,404.00$

monthly compounding (n=12) $A = 1000\left(1 + \frac{0.04}{12}\right)^{12 \cdot 1} = \$10,407.42$

daily compounding $(n = 365)$ $A = 1000\left(1 + \frac{0.04}{365}\right)^{365} = \$10,408.08$

Seems as we increase n, the value goes up.
$10400 \to 10404 \to 10407 \to 10408$

As we compound more and more, the exponent goes up but the rate per period goes down, so we don't end up with an infinite number of dollars.

A particular expression that's important here:

$1\left(1 + \frac{1}{n}\right)^{n \cdot 1}$, $1 =$ investment of 1 dollar

$1 = interest$ rate of 100%

$n =$ number of times we compound

$t = 1$

Invest 1 dollar, for 1 year, at 100% interest, and see what happens as the number of compounding periods goes to infinity.

$n=1: \quad 1(1+1/1)^1 = 1(1+1)^1 = 1(2)^1 = 2$

2.25-2=.25

$n=2: \quad 1(1+1/2)^2 = 2.25 \Leftarrow$ not 4

$n=5: \quad 1(1+1/5)^5 = 2.48832 \Leftarrow$ bigger but not so much bigger

$n=10: \quad 1(1+1/10)^{10} = 2.5937 \Leftarrow$ bigger but not so much bigger

$n=10,000: \quad 1(1+1/10000)^{10000} = 2.718145927$

$n=100,000: \quad 1(1+1/100000)^{100000} = 2.718268237$

$$n=1,000,000: \quad 1\left(1+\frac{1}{1,000,000}\right)^{1,000,000} = 2.718280469$$

$$n=1,000,000,000: 1\left(1+\frac{1}{1,000,000,000}\right)^{1,000,000,000} = 2.718281827$$

the first 2 repeats
last two values:
2.718145927
2.718268237
2.718 is repeating

As $n \to \infty$, $1\left(1+\frac{1}{n}\right)^n \to 2.718\ldots$

$2.718\ldots$ is called e

Euler(oiler)

If we do a bit more mathmagic , we can produce the continous compounding formula

Imagine we compute the interest at the end of each tiny fraction of a second...we get a new formula:
$A = Pe^{rt}$

discrete compounding: $A = P\left(1+\frac{r}{n}\right)^{nt}$, *cont.* compounding= Pe^{rt}, where e is as defined above.

example 4: The amount A that results from investing a principal P of \$10,000 at an annual rate r of 4% compounded cont. for a time t of 1 year is

$A = 10000\,e^{0.04\cdot 1} = \$10,408.11$ This is the most we can get at 4% if we invest 10,000 for 1 year.

Interest= $A-P = \$10,408.11 - \$10,000 = \$408.11$

Example 5: On January 1, 2010, 2000 was placed in an IRA(individual retirement account) that will pay interest of 3% per annum compounded cont.

(a) What will the IRA be worth on Jaunary 1, 2030? t=2030-2010=20 *years*

A= $2000\cdot e^{0.03\cdot 20} = 3644.24$

Interst= $A-P = 3644.24 - 2000 = 1644.24$ (average over 20 years $1644.24/20 =$ 82 dollars per year average)

(b) What is the effective annual rate of interest?

b/c it says annual, we mean t=1:
interest= Prt
$A = 2000\,e^{0.03\cdot 1} = 60.91$

effective annual simplest interest rate
$60.91 = 2000\,r\cdot 1$ $\quad Interest = Prt$ (simple interest)

$$\frac{60.91}{2000} = r$$

$r = 0.0305 = 3.05\%$ little bit different from 3%.

If we compounded once per year , we'd have to use 3.05% and not 3% to get the same interest of 60.91.

$I = 2000\cdot 0.0305$

$$\frac{60.91}{2000} \approx 0.0305$$

Effective annual rate is the rate that gives the same amount of interest as the cont. comounding result, but the interest is computed only once.