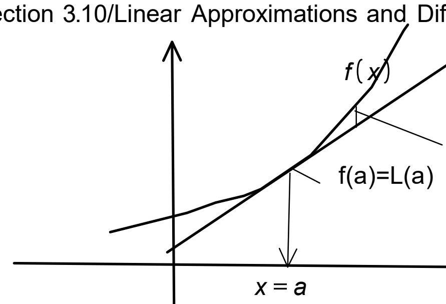
Please keep phones and computers away while doing notes.

Section 3.10/Linear Approximations and Differentials



$L(x)=f(a)+f'(a)(x-a) \Leftarrow$ L is the linearization of f at x=a.

$\xrightarrow{\text{based on}} y=b+m\cdot x$ $\quad \xrightarrow{1.} L(a)=f(a)+f'(a)(a-a)=f(a)$

$\xrightarrow{\text{2 slope at x=a}} f'(a)$ slope at a of both f and L.

Example 1/Page 251 : Find the linearization of the function f(x)= $\sqrt{x+3}$ at a=1(x=1) and use it to find the approximate value s of $\sqrt{3.98}$ and $\sqrt{4.05}$

$L(x)=f(a)+f'(a)(x-a)$

$f(1)=\sqrt{1+3}=\sqrt{4}=2,\quad f'(x)=\frac{d}{dx}\sqrt{x+3}=\frac{d}{dx}(x+3)^{1/2}$

$=\frac{1}{2}(x+3)^{-1/2}\frac{d}{dx}(x+3)=\frac{1}{2(x+3)^{1/2}}\cdot 1=\frac{1}{2\sqrt{x+3}}$

pick up here: $f'(1)=\frac{1}{2\sqrt{1+3}}=\frac{1}{2\sqrt{4}}=\frac{1}{2\cdot 2}=\frac{1}{4}$

$L(x)=2+\frac{1}{4}(x-1)=2+\frac{1}{4}x-\frac{1}{4}=\frac{8}{4}+\frac{1}{4}x-\frac{1}{4}=\frac{1}{4}x+\frac{7}{4}$

form: $mx+b$

So $\frac{1}{4}x+\frac{7}{4}\approx\sqrt{x+3}$ when x is close to x=1! Let's use this: $\sqrt{3.98}=\sqrt{0.98+3}$

Now $\sqrt{3.98}=\sqrt{0.98+3}=\frac{1}{4}(0.98)+\frac{7}{4}=1.995$

next..... Approximate $\sqrt{4.05}=\sqrt{1.05+3}=\frac{1}{4}(1.05)+\frac{7}{4}=2.0125$

$\sqrt{3.75}$ doesn't mean input 3.75..it means $\sqrt{0.75+3}=\frac{1}{4}(0.75)+\frac{7}{4}$ $\quad\sqrt{x+3}$

$\sqrt{x+3}$ = some number...

error= $|\sqrt{3.98}-1.995|=6.26\cdot 10^{-6}$ (tiny difference)

error for $x=1.05$: $=|\sqrt{4.05}-2.0125|=3.88\cdot 10^{-5}$ (tiny error)

error for x=2: $\sqrt{2+3}=\sqrt{5}\approx\frac{1}{4}(2)+\frac{7}{4}=2.25$

$\sqrt{5}=2.24$

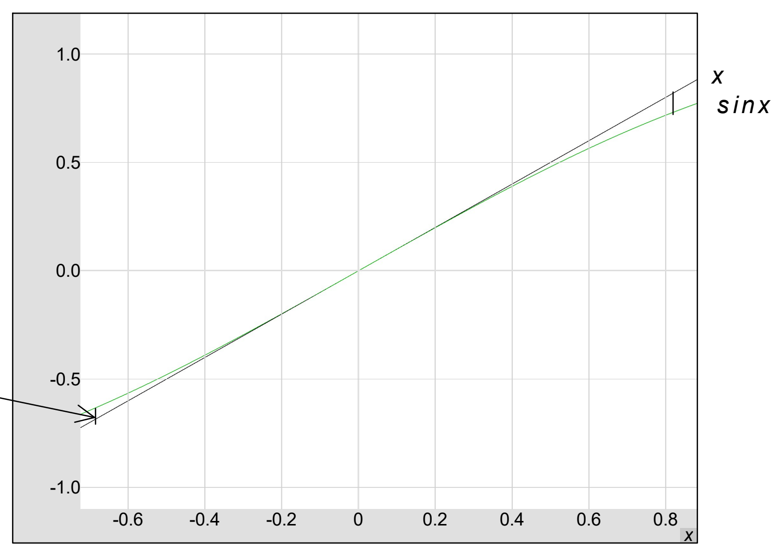
"Famous physics application" L(x) to represent sin(x) at x=0:

$L(x)=f(0)+f'(0)(x-0)=sin(0)+cos(0)(x-0)=0+1(x)=x$

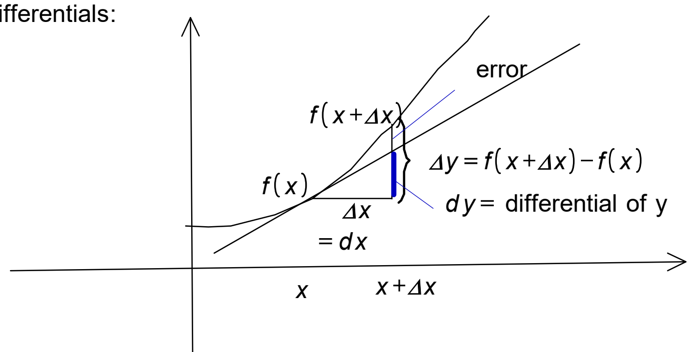
So when x is close to 0, we can represent sin(x) with just x!

Graphs shows that around x=0, sin(x) ≈ x , since the graph lines pretty much match!

To avoid signs, just do error= $|f(x)-L(x)|$ = magnitude of error big or small??



Differentials:



Usually $\frac{dy}{dx}$ is seen as ONE, UNBREAKABLE symbol!

$\frac{dy}{dx}=f'(x)$.. except here we treat dy/dx like a fraction:

$\frac{dy}{dx}=f'(x)\xrightarrow{\text{multiply by dx}} dy=f'(x)dx$

$dy\approx\Delta y$

$f'(x)dx\approx f(x+\Delta x)-f(x)$

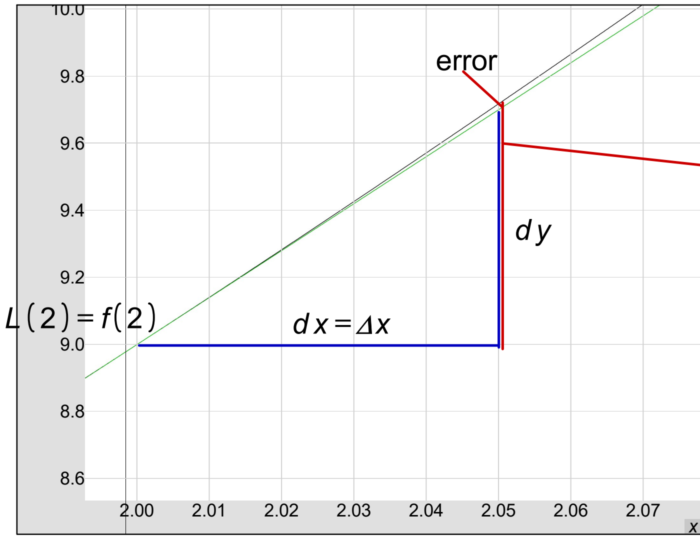
Example 3/Page 253: Compare the values of $\Delta y and$ dy if f(x)= x^3+x^2-2x+1 and x changes from x=2 to x=2.05:

$dx=\Delta x=2.05-2.00=0.05,\quad f'(x)=(x^3)'+(x^2)'+(-2x)'+(1)'=3x^2+2x-2$

$dy=f'(x)dx$: $dy=(3x^2+2x-2)dx\xrightarrow{\text{plug in the values}} dy=(3\cdot 2^2+2\cdot 2-2)(0.05)=0.7$

exact change in f as x goes from x=2 to x=2.05: $f(2.05)-f(2.00)=2.05^3+2.05^2-2\cdot 2.05+1-(2^3+2^2-2\cdot 2+1)=0.717625$

$dy=0.7,\ \Delta y=0.717625$ ... so dy ≈ Δy



$$L(x) = f(2) + f'(2)(x-2)$$
$$= 2^3 + 2^2 - 2\cdot 2 + 1 + (3\cdot 2^2 + 2\cdot 2 - 2)(x-2)$$
$$= 9 + 14(x-2)$$

Δy Picture shows that dy and Δy are very close but not identical. We express this by writing $dy \approx \Delta y$

Example 4/Application:
The radius of a sphere was measured to be 21 cm with a possible error in measurement of at most .05cm. What is the maximum error in using this value of the radius to compute the volume fo this sphere?
$r =$ radius, Δr=change in radius= $0.05\,cm$

V= $\frac{4}{3}\pi r^3$ (volume of sphere)

$$\frac{dV}{dr} = \frac{4}{3}\pi\cdot 3\cdot r^2 = 4\pi r^2 \xrightarrow{\text{solve for dV}} dV = 4\pi r^2\, dr$$

What is $\Delta V =$ exact change in the volume
dV

$r = 21$ ($dr = \Delta r = 0.05$)

dV= $4\cdot\pi\cdot 21\cdot 0.05^2 \approx 277\,\text{cm}^3$. Seems like a big number but lets compare to the volume at r=21:
$\Delta V =$ exact change $\approx$ dV=277cm^3

relative error using ΔV = $$\frac{\Delta V}{V} \xrightarrow{\text{because } \Delta V \approx dV} \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^{3-2}} = \frac{4\,dr}{\frac{4r}{3}} = \frac{4\,dr}{1}\cdot\frac{3}{4r} = 3\frac{dr}{r}$$

In words: The relative error in volum is 3 times the relative error in the radius : $\frac{dr}{r} \Leftarrow$ relative error in radius

relative error in volume $\approx 3\cdot\frac{0.05}{21} = 0.0071 \xrightarrow{\text{in percent form}} 0.71\% \Leftarrow$ small!!

.71%= $\frac{0.71}{100} \Leftarrow$ divide the volume into 100 boxes..the change in the volume when x goes from r=21 to r=21.05 is .71 out of 100, which is not even one box!

"calculus "= Latin for "small stone"