Calculus 1 Notes 11 28 2023/Section 4.4:
Indeterminate Forms and L'Hospital's Rule:

Please be sure to take very detailed notes.

$F(x)=\frac{\ln x}{x-1}$ $\quad F(1)=\frac{\ln(1)}{1-1}=\frac{0}{0}\Leftarrow$ undefined at x=1

Behavior around x=1? $\lim\limits_{x\to 1}\frac{\ln x}{x-1}$ direct. sub fails.

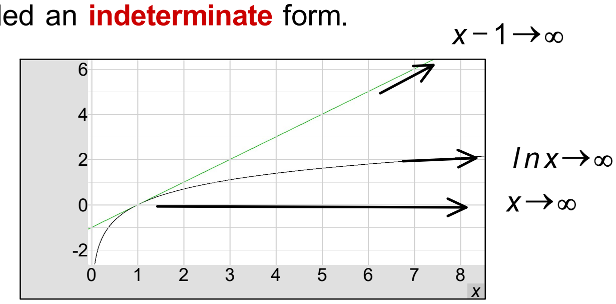
1. rationalize ..b/c it's an ln and x-1 fails too..

recall: $\frac{\sqrt{x+1}-1}{x}\Leftarrow$ rationalization worked here...

b/c we do $\frac{\sqrt{x+1}-1}{x}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$

We need a new process..besides trying graphically ..

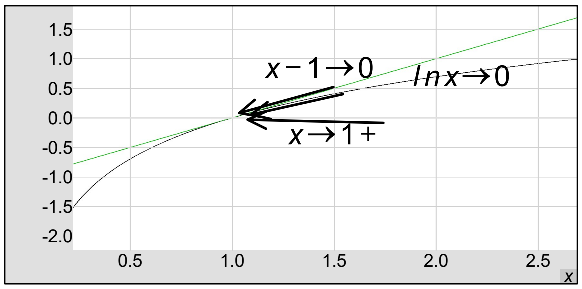
process here: When we have the form $\frac{0}{0}$, this is called an **indeterminate** form.



In the form $\frac{\ln x}{x-1}$, as x goes to 1 , $\ln 1.00001 \Rightarrow 0$

$\lim\limits_{x\to\infty}\frac{\ln x}{x-1}$. $\lim\limits_{x\to\infty}\ln x\to\infty$, $\quad\lim\limits_{x\to\infty}(x-1)\to\infty$

form $\frac{\infty}{\infty}\Leftarrow$ indeterminate form



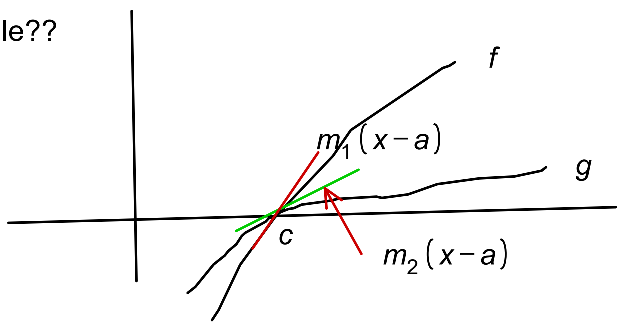
Suppose f and g are differentiable functions and $g'(x)\neq 0$ on an open interval I e.g:(a,b), (e.g. stands for exampli grati) and this interval contains x=c . (f and g don't have to be differentiable at x=c)

$\lim\limits_{x\to a} f(x)=0$, $\quad\lim\limits_{x\to a} g(x)=0$ $\qquad\Rightarrow\frac{0}{0}$ or $\frac{\infty}{\infty}$

or $\lim\limits_{x\to a} f(x)=\infty$, $\lim\limits_{x\to a} g(x)=\infty$

then $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}\Leftarrow$ not the quotient rule!!

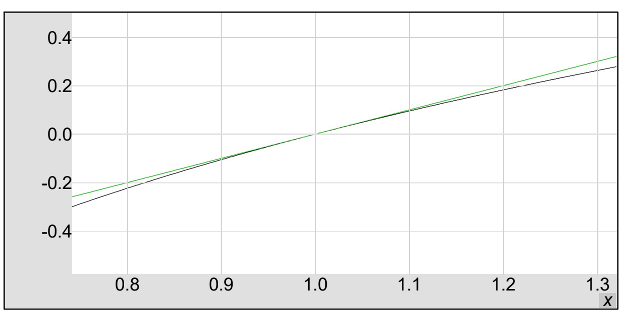
Why is this reasonable??



ratio of tangent lines: $\frac{m_1(x-a)}{m_2(x-a)}=\frac{m_1}{m_2}$

ratio of slopes

Behavior of tangent lines and graphs are very similar around x=c.



around x=1:
$\ln x$ is like the black straight line
x-1 is already a straight line
to find $\lim\limits_{x\to 1}\frac{\ln x}{x-1}$ we can use the tangent lines, so we get $\lim\limits_{x\to 1}\frac{m_1(x-1)}{m_2(x-1)}=\frac{m_1}{m_2}$

example 1: find $\lim\limits_{x\to 1}\frac{\ln x}{x-1}$, $\quad$ check wit x=1: $\frac{\ln 1}{1-1}=\frac{0}{0}$ undefined..indeterminate form

differentiate each function indepdendently... $\lim\limits_{x\to 1}\frac{1/x}{1}=\lim\limits_{x\to 1}\frac{1}{x}=\frac{1}{1}=1$

example 2: $\lim\limits_{x\to\infty}\frac{e^x}{x^2}$ $\qquad$ direct sub: $\frac{e^\infty}{\infty^2}=\frac{\infty}{\infty}$ (indeterminate form)

$\lim\limits_{x\to\infty}\frac{e^x}{2x}\Leftarrow$ take derivatives of top and bottom, no quotient rule

try direct sub again: $\frac{e^\infty}{2\cdot\infty}=\frac{\infty}{\infty}$ (still indeterminate)

$\lim\limits_{x\to\infty}\frac{e^x}{2}\Leftarrow$ derivative of top and bottom, no quotient rule

$\frac{e^{\infty}}{2} = \frac{\infty}{2} = \infty \Leftarrow$ *this* is a valid limit..stop here!!

example 3: $\lim_{x\to\infty} \frac{ln\,x}{\sqrt[3]{x}}$ try direct sub: $ln(\infty) = \infty$

form: $\frac{\infty}{\infty}$

$ln(\infty) \Leftarrow$ *not* strictly correct math, but communicates the correct idea of plugging in ever bigger values of x!

$\sqrt[3]{x} \Rightarrow$ plug in " ∞ ", $\sqrt[3]{\infty} = \infty$

$\sqrt[3]{10000000000} = 2154$

$\sqrt[3]{1000000000000} = 10000$

$\lim_{x\to\infty} \frac{1/x}{\frac{d}{dx}(x)^{1/3}}$

$\lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{1/3-1}}$

$\lim_{x\to\infty} \frac{1/x}{1/3 \cdot x^{-2/3}} \Leftarrow$ here..helps to use KCF on fraction

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}} = \frac{1}{1/3} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x^{2/3}}} = 3 \lim_{x\to\infty}\left(\frac{1}{x}\,\frac{x^{2/3}}{1}\right) = 3 \lim_{x\to\infty} x^{2/3-1} = 3\cdot \lim_{x\to\infty} x^{2/3-3/3}$$

$$= 3 \lim_{x\to\infty} x^{-1/3}$$

$$= 3\cdot \lim_{x\to\infty} \frac{1}{\sqrt[3]{x}} = 3\cdot 0 = 0$$

We have only two forms: $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$2\cdot 3 = \frac{2}{1/3} = \frac{2/1}{1/3} = \frac{2}{1}\cdot\frac{3}{1} = 6$

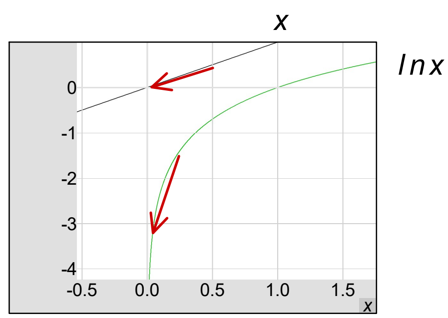
product of two functions: $f\cdot g = \frac{f}{1/g} \Leftarrow$ by k c f

$f\cdot g = \frac{g}{1/f} \Leftarrow$ by gcf

$\Rightarrow \frac{f}{\frac{1}{g}} = \frac{\frac{f}{1}}{\frac{1}{g}} = \frac{f}{1}\cdot\frac{g}{1} = fg$

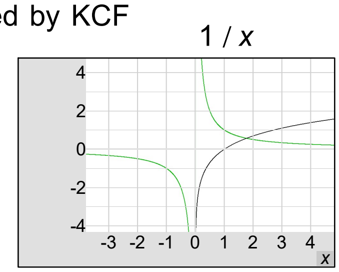
$\lim_{x\to 0+} x\,ln\,x$ right it's $0\cdot ln0$, $ln0$ is not defined $0\cdot$ undefined??

$0\cdot(-\infty)$ (doesn't mean we ever reach 0 or $-\infty$), $\underline{0.00000001\cdot -100000000000}$



$\lim_{x\to 0+} \frac{ln\,x}{1/x}$ allowed by KCF

$\frac{-\infty}{\infty} \Rightarrow$ graphs!



L'Hopital's:

$\lim_{x\to 0+} \frac{1/x}{-1x^{-1-1}}$

$\lim_{x\to 0+} \frac{1/x}{-1x^{-2}} \Leftarrow$

$\lim_{x\to 0+} \frac{1/x}{-1/x^2} \Rightarrow \text{kcf} \Rightarrow \lim_{x\to 0+}\left(\frac{1}{x}\cdot\frac{-x^2}{1}\right) = \lim_{x\to 0+} \frac{-x^2}{x} = \lim_{x\to 0+}(-x) = -0 = \mathbf{0}$

$\frac{0}{0}$

$\frac{\infty}{\infty}$

$0(-\infty)$

0^0

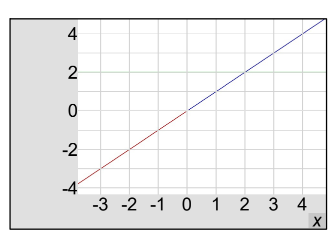
example 9: $\lim_{x\to 0+} \boldsymbol{x^x} \Rightarrow$ direct sub first: 0^0...migth say =0 (*no*)

$0.00001^{0.00001} \to 0.0000001^{0.0000001}$

for x>0, x= e^{lnx}

recall $2 = e^{ln2}$, $x = e^{lnx}$

$2 = e^{ln2}$



$\lim_{x\to 0+} e^{lnx\cdot x}$ $e^{f(x)}$ is cont, so we can put limit in top

$$e^{\lim_{x\to 0+}(lnx\cdot x)} = e^{\lim_{x\to 0+}\frac{lnx}{1/x}} = e^{\mathbf{0}} = 1$$

$\lim_{x\to\infty} x^{1/x}$ x in the base: $x\to\infty$, $x\to\infty$

$\frac{1}{x}$: $x\to\infty$, $\frac{1}{x}\to 0$ $\Rightarrow \infty^0$

$x = e^{lnx}$, $x > 0$

$\lim_{x\to\infty} e^{lnx\cdot\frac{1}{x}}$ (rewrite x as e^{lnx})

$e^{\lim_{x\to\infty}\left(lnx\cdot\frac{1}{x}\right)}$ (put limit into exponent b/c e is cont.)

analyze exponent: $\frac{lnx}{x}$ $x\to\infty$, $lnx\to\infty$; $x\to\infty$, $x\to\infty$ recall: $2\cdot\frac{1}{4}=\frac{2}{4}$

$e^{\lim_{x\to\infty}\frac{lnx}{x}}$

$\Downarrow$ L'Hopital's or $L'Hospital's$

$e^{\lim_{x\to\infty}\frac{1/x}{1}}$

direct sub: $e^{\lim_{x\to\infty}\frac{1}{x}} = e^0 = 1$

summary: $\frac{0}{0}$ or $\frac{\infty}{\infty}$ $\Leftarrow$ use L'Hopitals only!

if not, try to rewrite $f\cdot g = \frac{f}{1/g}$, or $f\cdot g = \frac{g}{1/f}$

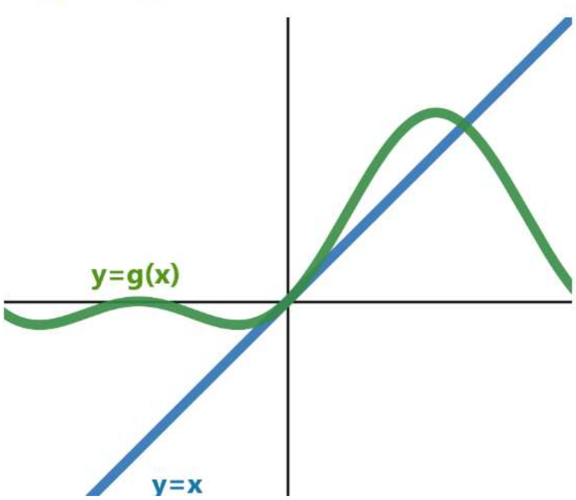
takes forms like $0(\infty)$ and turns them into $\frac{\infty}{\infty}$

When we say $0\cdot\infty$, don't take this to literally mean "zero"$\cdot$ actual infinity...
one is going to 0 and the other is going to infinity

$0.00001\cdot 10000000000$ (tiny number$\cdot$ massive number)

do not use the quotient rule.... $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

The graph of a function $g(x)$ and its tangent line at $x=0$ are shown. Find the exact value of $\lim_{x\to 0}\frac{g(x)}{38\cdot e^x - 38}$.



$\lim_{x\to 0}\frac{g(x)}{38e^x - 38}$

direct sub:

$\frac{g(0)}{38e^0 - 38} = \frac{0}{0}$ undefined

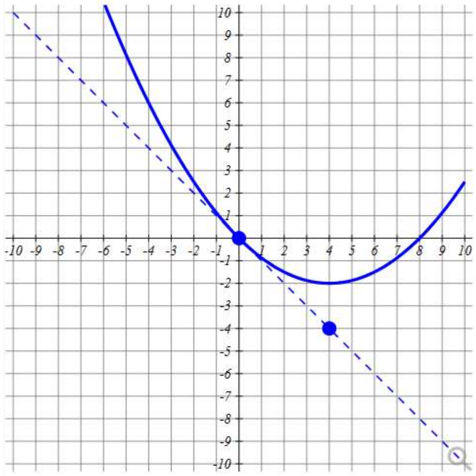
L'Hopital's:

$\lim_{x\to 0}\frac{g'(x)}{38e^x}$

direct sub now: $\frac{g'(0)}{38e^0}$

slope of tangent line top

$\frac{1}{38}$



Find

$\lim_{x\to0} \frac{f(x)}{x} =$ −1 -1

$\lim_{x\to0} \frac{f(x)}{3e^x - 3} =$ −1 / 3 $-\frac{1}{3}$

te

$\lim_{x\to0} \frac{f(x)}{x}$

notice that $\lim_{x\to0} f(x) = 0$ from solid curved line

the limit of x as x goes to 0 is also 0.

so we have $\frac{0}{0}$

differentiate: $\lim_{x\to0} \frac{f'(x)}{1} = \lim_{x\to0} f'(x)$

direct sub= $f'(0) =$ slope of tangent line at x=0
=-1

$\lim_{x\to0} \frac{f(x)}{3e^x - 3} \quad \frac{f(0)}{3e^0 - 3} = \frac{0}{0} \Leftarrow$ undefined

L'Hopital: $\lim_{x\to0} \frac{f'(x)}{3e^x}$

direct sub= $\frac{f'(0)}{3e^0} = \frac{-1}{3}$ (-1 from slope of tangent line *at* x=0..dashed blue line)