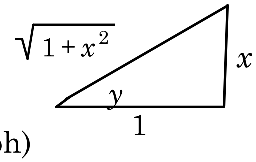
Back in section 3.5, we have derivatives of inverse trig functions.

Find y' *for* $y = tan^{-1}(x) = \tan^{-1}\left(\frac{x}{1}\right)$



$\sqrt{1^2+x^2} = \sqrt{1+x^2}$

$\Leftarrow \tan(y) = \frac{x}{1}$

$y = tan^{-1}(x)$

$tan(y) = x \Leftarrow$ why this step? (based on graph)

implicit: $sec^2(y)\ y' = 1$ (LHS has chain rule y means y(x))

$$y' = \frac{1}{sec^2(y)}$$

$$sec(y) = \frac{1}{cos(y)} = \frac{1}{\ }$$

$$\frac{d}{dx} tan^{-1}(x) = \frac{1}{1+x^2}$$

$$y' = \frac{1}{[secy]^2} \Leftarrow \text{refer to graph again...}$$

$$\frac{d}{dx} tan^{-1}(u(x))$$

$$y' = \frac{1}{\left(\frac{1}{cosy}\right)^2}$$

from figure

$u(x)$ inside, $\tan^{-1}$ (u) outside

$$y' = \frac{1}{\frac{1}{cos^2 y}} \rightarrow KCF \rightarrow y' = cos^2 y = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{(\sqrt{1+x^2})^2} = \frac{1}{1+x^2}$$

example: $\frac{d}{dx} tan^{-1}(2x) = \frac{1}{1+(2x)^2}(2x)' = \frac{1}{1+2^2x^2}(2) = \frac{2}{1+4x^2}$

$\frac{d}{dx} \boldsymbol{tan^{-1}}(1/2x)$ (inside= $1/2x$, $outsie = \tan^{-1}$ (...))

$$= \frac{1}{1+(1/2x)^2}\left(\frac{1}{2}x\right)' = \frac{1}{1+\frac{1}{2^2}x^2}\left(\frac{1}{2}\right) = \frac{1}{1+\frac{1}{4}x^2}\cdot\frac{1}{2} = \frac{1}{1\cdot 2+\frac{1}{4}\cdot 2\cdot x^2} = \frac{1}{2+\frac{1}{2}x^2} \text{ (one form)}$$

$\frac{d}{dx} tan^{-1}(x+1)$, inside= $x+1$, outside= $tan^{-1}(\ldots)$

$$= \frac{1}{1+(x+1)^2}[x+1]' = \frac{1}{1+(x+1)^2}\cdot 1 = \frac{1}{1+(x+1)^2}$$

$\frac{d}{dx} tan^{-1}(u(x)) = \frac{1}{1+[u(x)]^2}u'(x)$ (chain rule and rule for inverse tangent)

$$= \frac{u'(x)}{1+u^2(x)}$$

f(x)= $x \cdot tan^{-1}(x)$ (product)　　$(uv)' = u'v + uv'$

$$f'(x) = (x)'\ tan^{-1}(x) + x\ [tan^{-1}(x)]'$$

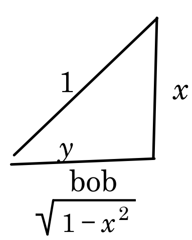
$$= 1\,tan^{-1}(x) + x \cdot \frac{1}{1+x^2}$$

$$= tan^{-1}(x) + \frac{x}{1+x^2}$$

Another one: $\frac{d}{dx} sin^{-1}(x)$

$y = sin^{-1}(x) = sin^{-1}\left(\frac{x}{1}\right)$

$sin(y) = x \Rightarrow$ look at picture!!



$1^2 = x^2 + bob^2$

$1 = x^2 + bob^2$

$1 - x^2 = bob^2$

$\sqrt{1-x^2} = bob$

So we can evaluate

$y'(x) = \frac{1}{\sqrt{1-x^2}}$ only

over $(-1, 1)$.

implict: $cos(y)\ y' = 1$

$$y' = \frac{1}{cosy} = \frac{1}{\sqrt{1-x^2}}$$

$$y'(1) = \frac{1}{\sqrt{1-1^2}} = \frac{1}{\sqrt{1-1}} = \frac{1}{\sqrt{0}} = \frac{1}{0} \text{ DNE!}$$

$$y'(-1) = \frac{1}{\sqrt{1-(-1)^2}} = \frac{1}{\sqrt{1-1}} = \frac{1}{0} \text{ DNE!!}$$

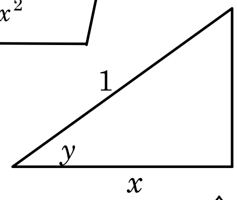
$$y'(2) = \frac{1}{\sqrt{1-(2)^2}} = \frac{1}{\sqrt{1-4}} = \frac{1}{\sqrt{-3}}$$

$$= \frac{1}{\sqrt{3}\,i} \Leftarrow \text{doesn't work here!!}$$

$\frac{d}{dx} sin^{-1}(3x)$, inside=3x, outside= $sin^{-1}(\ldots.)$

$= \frac{1}{\sqrt{1-(3x)^2}}(3x)' = \frac{1}{\sqrt{1-3^2x^2}} \cdot 3 = \frac{1 \cdot 3}{\sqrt{1-9x^2}} = \frac{3}{\sqrt{1-9x^2}}$

$\frac{d}{dx} cos^{-1}(x)$ $\quad y = cos^{-1}(x) = cos^{-1}\left(\frac{x}{1}\right)$



$\frac{blah}{\sqrt{1-x^2}}$

blah ^{2} +x ^{2} =1 ^{2}

blah ^{2} =1-x ^{2}

blah= $\sqrt{1-x^2}$

$cos(y) = \frac{x}{1} = x$

implicit: $-sin(y)\; y' \;=\; 1$

$y' = \frac{1}{-sin(y)} = -\frac{1}{\sqrt{1-x^2}}$

$\Uparrow$ in picture it's clear

that $\underbrace{sin(y) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}}_{\text{trig!}}$

example: $\frac{d}{dx} \boldsymbol{cos}^{-1}(2x) = -\frac{1}{\sqrt{1-(2x)^2}}(2x)' = -\frac{1}{\sqrt{1-2^2x^2}} \cdot 2 = -\frac{1 \cdot 2}{\sqrt{1-4x^2}} = -\frac{2}{\sqrt{1-4x^2}}$

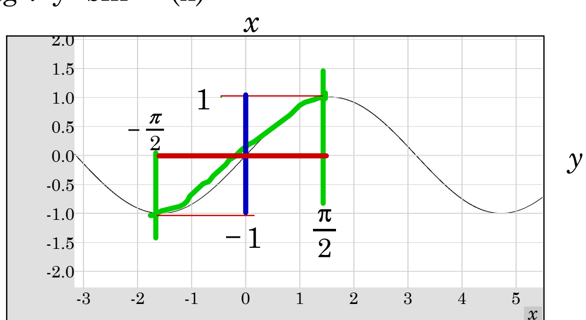
deeper look at $\frac{d}{dx} cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$ good over $(-1, 1)$ only

Why does i exist?
Read the story of
Tartaglia and Bombelli!

y'(1)= $-\frac{1}{\sqrt{1-(1)^2}} = -\frac{1}{\sqrt{1-1}} = -\frac{1}{\sqrt{0}} = -\frac{1}{0} \Leftarrow$ not defined!

$y'(-1) = -\frac{1}{\sqrt{1-(-1)^2}} = -\frac{1}{\sqrt{1-1}} = -\frac{1}{\sqrt{0}} = -\frac{1}{0} \Leftarrow$ not defined!

inverse trig : y=sin $^{-1}$ (x)



Over the green part, we're one-to-one.

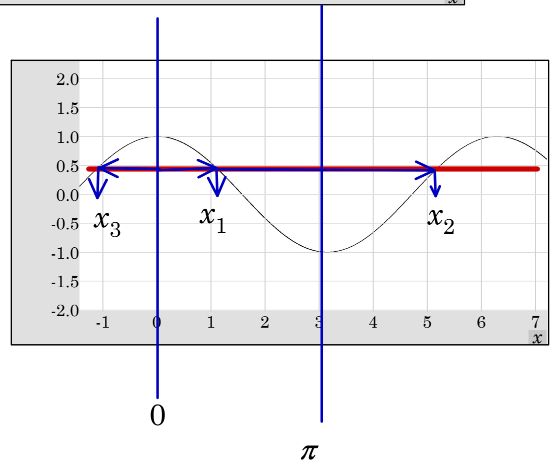
Now read graph from vertical to horizontal..that's the inverse sine function.

domain= $[-1, 1]$ $\Leftarrow$ vertical

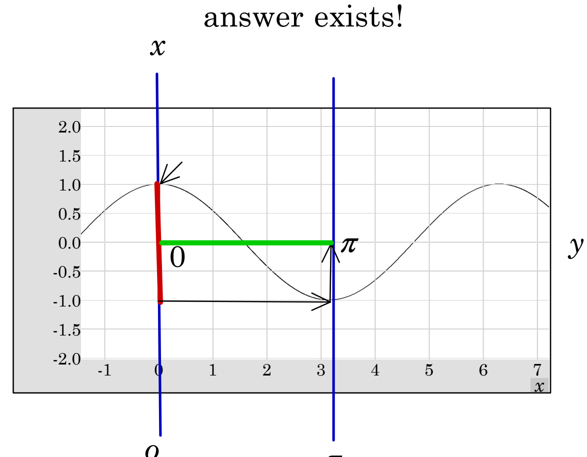
range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Leftarrow$ horizontal

$y'(x) = \frac{1}{\sqrt{1-x^2}}$

$y = cos^{-1}(x)$



$y = 1/2$ ..does it come from x_1 or x_2 or x_3..no unique answer exists!



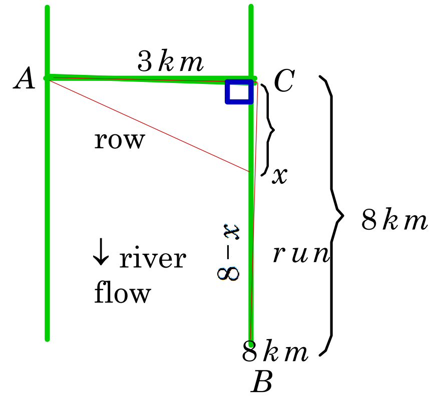
$\frac{d}{dx} cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$ domain is $[-1, 1]$

range is $[0, \pi]$

Drawing two separate graphs for the function and inverse..is not really needed except for the purposes of convention where the indepdenent variable is always on the horizontal axis!

Example $4/Page$ 329:

A person launches a boat from point A on a bank ofa straight river, 3km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible.



more info: person can run at 8 km /hr and row at 6 km/hr. Jump out of boat at x and run!

Since 8 the total, 8-x is the balance.

distance rowed: $\sqrt{x^2+3^2} = \sqrt{x^2+9}$

distance run: $8-x$

$d =$ speed $\cdot$ time ↑ distance in terms of x, speeds are given!

$$\sqrt{x^2+9} = 6\cdot t \Rightarrow \frac{\sqrt{x^2+9}}{6} = t \text{ (time we row)}$$

$$\underset{d}{8-x} = \underset{r}{8}\cdot \underset{t}{t} \rightarrow \frac{8-x}{8} = t \text{ (time we run)}$$

Find the mininimum time:

$$\frac{\text{stuff}}{a} = \frac{1}{a}\cdot \text{stuff}$$

total time to row and run: $T(x) = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}$ (add up times)

$T(x)$ (includes time to run and row ..depends on x..we decide where we land and jump out!)

$$T'(x) = \frac{1}{6}\left(\frac{1}{2}(x^2+9)^{-1/2}(x^2+9)'\right) + \frac{1}{8}(8-x)'$$

$$= \frac{1}{6}\left(\frac{1}{2}\cdot\frac{1}{\sqrt{x^2+9}}(2x)\right) + \frac{1}{8}(-1) = \frac{1}{\cancel{12}}\frac{\cancel{2}x}{\sqrt{x^2+9}} - \frac{1}{8} = \frac{x}{6\sqrt{x^2+9}} - \frac{1}{8}$$

$$T'(x) = 0$$

$$\frac{x}{6\sqrt{x^2+9}} - \frac{1}{8} = 0 \rightarrow \frac{x}{6\sqrt{x^2+9}} = \frac{1}{8} \rightarrow 8x = 6\sqrt{x^2+9} \rightarrow \text{divide by } 2 \rightarrow 4x = 3\sqrt{x^2+9}$$

$$\xrightarrow{\text{square}} (4x)^2 = \left(3\sqrt{x^2+9}\right)^2$$

$$4^2x^2 = 3^2\left(\sqrt{x^2+9}\right)^2$$

$$16x^2 = 9(x^2+9)$$

$$16x^2 = 9x^2 + 9(9)$$

$$16x^2 = 9x^2 + 81$$

$$7x^2 = 81$$

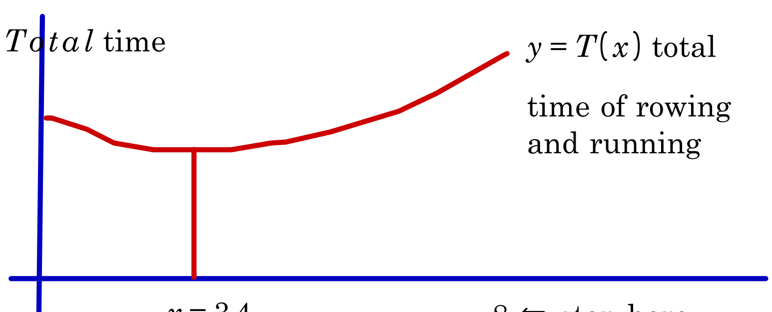
$$x^2 = \frac{81}{7} \rightarrow x = \pm\sqrt{\frac{81}{7}} = \frac{\pm\sqrt{81}}{\sqrt{7}} = \pm\frac{9}{\sqrt{7}}$$

$x =$ distance downstream:

$0 \le x \le 8$ (from picture)

$x = +\frac{9}{\sqrt{7}} = 3.4$

$x = -3.4$ (not in the range 0<=x<=8..not in domain of T)



So x=3.4 means get into your boat, row so as to land 3.4 km downstream, jump out and run the rest of the way!

$$= 2\cdot \underbrace{\tan^{-1}}_{\text{outisde}}(\underbrace{\sin(x)}_{\text{inside}}), \text{ then } y'(x) =$$ chain rule and rule for $\tan^{-1}$

c

$$y'(x) = [2\tan^{-1}(\sin(x))]'$$

$$= 2[\tan^{-1}(\sin(x))]' \text{ (2 outside)}$$

$$= 2\cdot\frac{1}{1+\sin^2 x}(\sin x)'$$

$$= \frac{2\cos x}{1+\sin^2 x}$$

$[\sin x]^2 = \sin^2 x$

$= 2\cdot\frac{1}{\sqrt{1-(\sin x)^2}}(\sin x)'$ (chain rule on $\tan^{-1}$)

$= 2\cdot\frac{1}{\sqrt{1-\sin^2 x}}(\cos x)$

$= \frac{2\cdot 1\cdot\cos x}{\sqrt{1-\sin^2 x}} = \frac{2\cos x}{\sqrt{1-\sin^2 x}}$

$= \underbrace{5\cos^{-1}}_{\text{outside}}\underbrace{(4x+11)}_{\text{inside}}$, then

$$\frac{d}{dx}\cos^{-1}(u(x)) = -\frac{1}{\sqrt{1-u^2(x)}}u'(x)$$

$$y' = [5\cos^{-1}(4x+11)]'$$
$$= 5[\cos^{-1}(4x+11)]' \text{ (5 outside)}$$
$$= 5\,\frac{-1}{\sqrt{1-(4x+11)^2}}(4x+11)'$$
$$= 5\,\frac{-1}{\sqrt{1-(4x+11)^2}}(4)$$
$$= \frac{5(-1)(4)}{\sqrt{1-(4x+11)^2}} = \frac{-20}{\sqrt{1-(4x+11)^2}}$$

If $f(x) = 4\cot^{-1}(x^5)$, then $\frac{dy}{dx} = [\;$

rule (not derived in class)
for $\frac{d}{dx}\cot^{-1}(u(x)) = -\frac{1}{1+u^2(x)}u'(x)$

inside= x^5, outside=cot^{-1} (...)

$f'(x) = [4\cot^{-1}(x^5)]'$

rule: $-\frac{a}{b} = \frac{-a}{b}$ or $\frac{a}{-b}$ but NOT $\cancel{\frac{-a}{-b}}$

$$= 4[\cot^{-1}(x^5)]'$$
$$= 4\left(-\frac{1}{1+(x^5)^2}(x^5)'\right) = 4\left(-\frac{1}{1+x^{5\cdot 2}}\right)\cdot 5x^4 = \frac{4}{1}\left(-\frac{1}{1+x^{\mathbf{10}}}\right)\cdot\frac{5x^4}{1} = \frac{4(-1)\cdot 5x^4}{1+x^{\mathbf{10}}} = \frac{-20x^4}{1+x^{10}}$$

or $-\frac{20x^4}{1+x^{10}}$ or $-\frac{20x^4}{x^{10}+1}$

On your own following our examples....

derive (using a little triangle...) $\frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2} \Leftarrow$ first big idea $\cot^{-1}\left(\frac{x}{1}\right)$