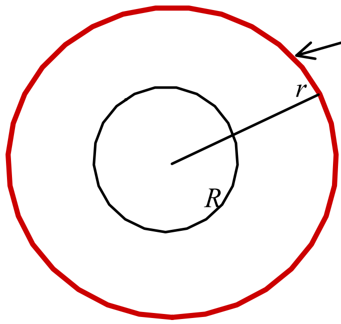
Given f entire, $\exists$ M, R>0, and an integer n $\geq$ 1, such that |f(z)| $\leq M|z|^n$ for $|z| > R$

Show that f is a polynomial of degree $\leq n$.

Since f is entire, meaning differentiable everywhere, it can be represented as a power series, so

$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \forall z \in \mathbf{C}$ where $a_n = \frac{f^n(0)}{n!}$, $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots a_n z^n + \ldots$ (terms here get cut off)



Fix $r > R$, for |z|=r>R , $|f(z)| \leq M r^n$

By Cauchy's Estimates , $\frac{|f^k(0)|}{k!} \leq \frac{M r^n}{r^k}$

So $\forall$ k $\geq n$, we have $|a_k| = \frac{|f^k(0)|}{k!} \leq \frac{M r^n}{r^k} = \frac{M}{r^{k-n}}$

$\lim_{k \to \infty} \frac{M}{r^{k-n}} = 0$, $\forall$ k $\geq n$

So f is a polynomial of degree $\leq n$

What this argument says is that all the coefficients after n vanish and so only the first n coefficients survive, giving us a polynomial.